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Domination Numbers On the Boolean Function Graph B($\overline{K_p}$, NINC, L(G))of a Graph

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Abstract: *For any graph G, let V(G) and E(G) denote the vertex set and edge set of G respectively. The Boolean function graph B($\overline{K}_p$, NINC, L(G)) of G is a graph with vertex set V(G)∪E(G) and two vertices in B($\overline{K}_p$, NINC, L(G)) are adjacent if and only if they correspond to two adjacent edges of G or to a vertex and an edge not incident to it in G. For brevity, this graph is denoted by $B_2(G)$. In this paper, domination number, independent, connected, total, cycle, point-set, restrained, split and non-split domination numbers in $B_2(G)$ are determined. Also the bounds for the above numbers are obtained.*

Keywords: *Boolean function graph, domination number*

1. Introduction

Graphs discussed in this paper are undirected and simple graphs. For a graph G, let V(G) and E(G) denote its vertex set and edge set respectively. For a connected graph G, the *eccentricity* $e_G(v)$ of a vertex v in G is the distance to a vertex farthest from v. Thus, $e_G(v) = \{d_G(u, v) : u \in V(G)\}$, where $d_G(u, v)$ is the distance between u and v in G. If there is no confusion, then we simply denote the eccentricity of vertex v in G as e(v) and d(u, v) to denote the distance between two vertices u, v in G respectively. The minimum and maximum eccentricities are the *radius and diameter* of G, denoted r(G) and diam(G) respectively. The *neighborhood* $N_G(v)$ of a vertex v is the set of all vertices adjacent to v in G. The set $N_G[v] = N_G(v) \cup \{v\}$ is called the closed neighborhood of v. A set S of edges in a graph G is said to be independent, if no two of the edges in S are adjacent. A set of independent edges covering all the vertices of a graph G is called *perfect matching*. An edge e = (u, v) is a *dominating edge* in a graph G, if every vertex of G is adjacent to at least one of u and v.

The concept of domination in graphs was introduced by Ore [15]. A set $D \subseteq V(G)$ is said to be a *dominating set* of G, if every vertex in V(G)−D is adjacent to some vertex in D. D is said to be a minimal dominating set if D−{u} is not a dominating set for any

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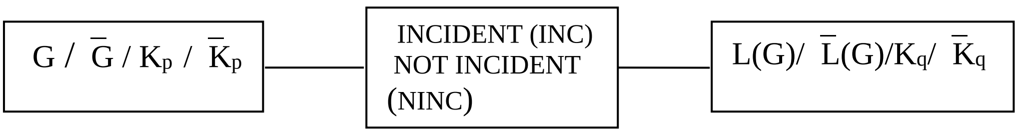
$u \in D$. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set. We call a set of vertices a γ-set, if it is a dominating set with cardinality $\gamma(G)$. Different types of dominating sets have been studied by imposing conditions on the dominating sets. A dominating set D is called a *connected* (*independent*) *dominating* set, if the induced subgraph <D> is connected (independent) [17]. D is called a total dominating set, if every vertex in V(G) is adjacent to some vertex in D [4]. A dominating set D is called a *cycle dominating set*, if the subgraph <D> has a Hamiltonian cycle and is called a *perfect dominating set*, if every vertex in $V(G)-D$ is adjacent to exactly one vertex in D [5]. D is called a *restrained dominating set*, if every vertex in $V(G)-D$ is adjacent to another vertex in $V(G)-D$ [6]. By γ_c, γ_i, γ_t, γ_0, γ_p and γ_r, we mean the minimum cardinality of a connected dominating set, independent dominating set, total dominating set, cycle dominating set, perfect dominating set and restrained dominating set respectively.

Sampathkumar and Pushpalatha [16] introduced the concept of point-set domination number of a graph. A set $D \subseteq V(G)$ is called a *point-set dominating set* (psd-set), if for every set $T \subseteq V(G)-D$, there exists a vertex $v \in D$ such that the subgraph $<T \cup \{v\}>$ induced by $T \cup \{v\}$ is connected. The point-set domination number $\gamma_{ps}(G)$ is the minimum cardinality of a psd-set of G. Kulli and Janakiram [14] introduced the concept of split and non-split domination in graphs. A dominating set D of a connected graph G is a *split* (*non-split*) *dominating set*, if the induced subgraph $<V(G)-D>$ is disconnected (connected). The split (non-split) domination number $\gamma_s(G)$ ($\gamma_{ns}(G)$) of G is the minimum cardinality of a split(non-split) dominating set. A set $F \subseteq E(G)$ is an *edge dominating set*, if each edge in E is either in F or is adjacent to an edge in F. The edge domination number $\gamma'(G)$ is the smallest cardinality among all minimal edge dominating sets. An edge dominating set $F \subseteq E(G)$ is an *independent edge dominating* (i.e.d) set, if the induced subgraph <F> is independent. The *independent edge domination number* $\gamma_i'(G)$ of G is the minimum cardinality of an i.e.d. set.

When a new concept is developed in graph theory, it is often first applied to particular classes of graphs. Afterwards more general graphs are studied. As, for every graph (undirected, uniformly weighted) there exists a adjacency (0, 1) matrix, we call the general operation as Boolean operation. Boolean operation on a given graph uses the adjacency relation between two vertices or two edges and incident relationship between vertices and edges and define new structure from the given graph. This adds extra bit information of the original graph and encode it new structure. If it is possible to decode the given graph from the encoded graph in polynomial time, such operation may be used to analyze various structural properties of original graph in terms of the Boolean graph. If it is not possible to decode the original graph in polynomial time, then that operation can be used in graph coding or coding of certain grouped signals. The mixed relations of incident, non-incident, adjacent and non-adjacent can be used to analyze nature of clustering of elements of communication networks. The concept of domination set can be visualized in each cluster as that cluster representatives and the domination set of whole network can be taken as representatives of entire network. If any clustering or a partition of vertices network such that each cluster having at least one representative or at least one element of dominating set of the given network

Whitney[19] introduced the concept of the line graph L(G) of a given graph G in 1932. The first characterization of line graphs is due to Krausz. The Middle graph M(G) of a graph G was introduced by Hamada and Yoshimura[7]. Chikkodimath and Sampathkumar[3] also studied it independently and they called it, the semi-total graph $T_1(G)$ of a graph G. Characterizations were presented for middle graphs of any graph, trees and complete graphs in [1]. The concept of total graphs was introduced by Behzad[2] in 1966. Sastry and Raju[18] introduced the concept of quasi-total graphs and they solved the graph equations for line graphs, middle graphs, total graphs and quasi-total graphs. These graphs are very much useful in the construction of various related networks from the underlying graphs of networks. This motivates us to define and study other graph operations. Using L(G), G, incident and non-incident, complementary operations, complete and totally disconnected structures, one can get thirty-two graph operations. As already total graphs, semi-total edge graphs, semi-total vertex graphs and quasi-total graphs and their complements (8 graphs) are defined and studied, we have studied all other similar remaining graph operations. This is illustrated below.



Here, $\overline{G}$ and L(G) denote the complement and the line graph of G respectively. K_p is the complete graph on p vertices.

The points and lines of a graph are called its elements. Two elements of a graph are neighbors, if they are either incident or adjacent. The *Total graph* T(G) of G has vertex set V(G)∪E(G) and vertices of T(G) are adjacent, whenever they are neighbors in G. The *Quasi- total graph*[18] P(G) of G is a graph with vertex set as that of T(G) and two vertices are adjacent if and only if they correspond to two nonadjacent vertices of G or to two adjacent edges of G or to a vertex and an edge incident to it in G. The *Middle graph* M(G) of G is one whose vertex set is as that of T(G) and two vertices are adjacent in M(G), whenever either they are adjacent edges of G or one is a vertex of G and the other is an edge of G incident with it. Clearly, E(M(G)) = E(T(G))−E(G).

The *Boolean function graph* B($\overline{K_p}$, NINC, L(G)) of G is a graph with vertex set V(G)∪E(G) and two vertices in B($\overline{K_p}$, NINC, L(G)) are adjacent if and only if they correspond to two adjacent edges of G or to a vertex and an edge not incident to it in G[]. For brevity, this graph is denoted by $B_2(G)$. In other words, $V(B_2(G)) = V(G) \cup V(L(G))$; and $E(B_2(G)) = [E(T(\overline{G})) - (E(\overline{G}) \cup E(\overline{L(G)}))] \cup E(L(G))$, where $\overline{G}$, L(G) and T(G) denote the complement, the line graph and the total graph of G respectively. The vertices of G and L(G) are referred as point and line vertices respectively.

In this paper, we determine the domination numbers mentioned above for this graph $B_2(G)$. The definitions and details not furnished in this paper may be found in [8].

2. Prior Results

In this section, we list some results with indicated references, which will be used in the subsequent main results. Let G be any (p, q) graph.

Theorem 2.1[16]: Let G = (V, E) be a graph. A set $S \subseteq V$ is a point-set dominating set of G if and only if for every independent set W in V−S, there exists a vertex u in S such that $W \subseteq N_G(u) \cap (V-S)$.

Observation[13]:

2.2: L(G) is an induced subgraph of $B_2(G)$ and the subgraph of $B_2(G)$ induced by point vertices is totally disconnected.

2.3: The number of vertices in $B_2(G)$ is p + q and if $d_i = deg_G(v_i)$, $v_i \in V(G)$ and the number of edges in $B_2(G)$ is $q(p - 3) + (1/2)\sum_{1 \leq i \leq p} d_i^2$.

2.4: The degree of a point vertex in $B_2(G)$ is q - $deg_G(v)$ and the degree of a line vertex e' is $deg_{L(G)}(e') + p - 2$. Also, if $d_2(v)$ denotes the degree of a point vertex v in $B_2(G)$, then $0 \leq d_2(v) \leq q$. Similarly, if $d_2(e')$ is the degree of a line vertex e' in $B_2(G)$, then $0 \leq d_2(e') \leq p + q - 3$.

Theorem 2.5[13]: $B_2(G)$ contains isolated vertices if and only if G is one of the following graphs: nK_1 and $K_{1,m} \cup tK_1$, for $n \geq 1$, $m \geq 1$ and $t \geq 0$.

Theorem 2.6[13]: $B_2(G)$ is disconnected (no component being K_1) if and only if $G \cong 2K_2$.

3. Main Results

In the following, we find the graphs G for which the domination number γ of $B_2(G)$ is 2 or 3.

Proposition 3.1: For any graph G having at least one edge, $\gamma(B_2(G)) \geq 2$.
Proof: Since $B_2(G)$ contains no vertex of degree $p + q-1$, the proposition follows.

Lemma 3.1: Any 2-set of $B_2(G)$ containing either two point vertices or a point vertex and a line vertex is not a dominating set of $B_2(G)$.
Proof: Let D be a 2-set of $B_2(G)$.
Case(i). D contains two point vertices.

Let $D = \{v_1, v_2\}$, where v_1, v_2 are any two point vertices in $B_2(G)$. Then $v_1, v_2 \in V(G)$. If v_1 and v_2 are not adjacent in G, then since G contains at least one edge and the subgraph of $B_2(G)$ induced by point vertices is totally disconnected, D is not a dominating set of $B_2(G)$. If v_1 and v_2 are adjacent in G, then the line vertex in $B_2(G)$ corresponding to the edge joining v_1 and v_2 is not adjacent to any of the vertices in D. Hence, D is not a dominating set of $B_2(G)$.
Case(ii): D contains one point and one line vertex.

Let e' be the line vertex in D and e be the corresponding edge in G. Then the point vertex in $B_2(G)$ corresponding to at least one of the end vertices of e is not adjacent to any of the vertices in D and hence D is not a dominating set of $B_2(G)$. Hence, the lemma follows.

Theorem 3.1: $\gamma(B_2(G)) = 2$ if and only if $\gamma_i'(G) = 2$, where $\gamma_i'(G)$ is the edge independent domination number of G.

Proof: Assume $\gamma(B_2(G)) = 2$. By Lemma 3.1., there exists a dominating set D of $B_2(G)$ containing exactly two line vertices. Let e_1 and e_2 be the edges in G corresponding to the line vertices in D. If e_1 and e_2 are adjacent in G, then D does not dominate the point vertex in $B_2(G)$ corresponding to the vertex in G common to both e_1 and e_2. Hence, e_1 and e_2 are nonadjacent edges in G. Since D dominates all the line vertices in $V(B_2(G))-D$, each line vertex in $V(B_2(G))-D$ is adjacent to at least one of the vertices in D. Thus, D is an independent dominating set of L(G) and hence $\gamma_i(L(G)) = 2$. That is, $\gamma_i'(G) = 2$. Converse is obvious.

Remark 3.1: D is also an independent dominating set of $B_2(G)$ and hence $\gamma_i(B_2(G)) = 2$ if and only if $\gamma_i'(G) = 2$.

Theorem 3.2: Let G be any graph having at least one edge. If $\gamma_i'(G) \neq 2$, then $\gamma(B_2(G))= 3$.
Proof: If $\gamma_i'(G) \neq 2$, then $\gamma(B_2(G)) \geq 3$ by Theorem 3.1. Let e = (u, v) be an edge in G, where u, v$\in$V(G). Let e' be the line vertex in $B_2(G)$ corresponding to e. Then u, v, $e' \in V(B_2(G))$. Then the set {u, v, e'} is a dominating set of $B_2(G)$. Thus, $\gamma(B_2(G)) \leq 3$. Hence, $\gamma(B_2(G)) = 3$.

Remark 3.2: D is also an independent dominating set of $B_2(G)$ and hence if $\gamma_i'(G) \neq 2$, then $\gamma_i(B_2(G)) = 3$.

Remark 3.3: If e_1 and e_2 are two adjacent edges and u is the common vertex, then the set containing the point vertex u and the line vertices corresponding to the edges e_1 and e_2 is a dominating set of $B_2(G)$.

The following propositions are stated without proof, as they are immediate from the definitions of related parameters.

Proposition 3.2: Let $G \neq K_{1,n} \cup mK_1$ and $C_3 \cup mK_1$, where $n \geq 1$ and $m \geq 0$. Then $\gamma(B_2(G)) \leq \alpha_1(G)$, where $\alpha_1(G)$ is the line covering number for G.

Proposition 3.3: If G is a graph other than a star and if $\gamma'(G) \geq 3$, then $\gamma(B_2(G)) \leq \gamma'(G)$, where $\gamma'(G)$ is the edge domination number of G.

Theorem 3.3: $B_2(G)$ has no dominating edge.
Proof: By Lemma 3.1., any 2-dominating set of $B_2(G)$ contains line vertices only. Therefore, if $B_2(G)$ has a dominating edge, then the end vertices must be line vertices and the corresponding edges in G are adjacent. But the point vertex corresponding to the vertex common to these edges is not adjacent to any of the above line vertices, which is a contradiction. Hence, $B_2(G)$ has no dominating edge.

Remark 3.4: Theorem 3.3., reveals that, $\gamma_c(B_2(G)) \geq 3$.

In the following, the graphs G for which the connected domination number $\gamma_c(B_2(G))$ is 3 are obtained.

Theorem 3.4: Let G be any graph such that $B_2(G)$ is connected. Then $\gamma_c(B_2(G)) = 3$ if and only if one of the following holds.

(i). There exists a dominating set D of L(G) with <D> $\cong P_3$ in L(G);

(ii). G contains $P_3 \cup K_1$ as an induced subgraph such that edges in G incident with the vertex in K_1 are adjacent to at least one of the edges of P_3; and

(iii). G contains $2K_2 \cup K_1$ as a subgraph such that edges incident with the vertex in K_1 are adjacent to at least one of the edges of $2K_2$.

Proof: Assume $\gamma_c(B_2(G)) = 3$. Then there exists a connected dominating set D having three vertices in $B_2(G)$ and the subgraph of $B_2(G)$ induced by D is either C_3 or P_3. If all the vertices of D are line vertices then $D \subseteq V(L(G))$ and hence L(G) contains C_3 or P_3 as an induced subgraph such that all the vertices in L(G) are adjacent to vertices of C_3 or P_3. If D contains two line vertices and one point vertex, then G contains $P_3 \cup K_1$ or $2K_2 \cup K_1$ as an induced subgraph such that edges incident with the vertex in K_1 are adjacent to at least one of the edges of P_3 or $2K_2$. Conversely, if (i), (ii) or (iii) holds, then there exists a connected dominating set D of $B_2(G)$ containing three vertices. Hence, $\gamma_c(B_2(G)) \leq 3$. But, $\gamma_c(B_2(G)) \geq 3$ by Remark 3.4. Thus, $\gamma_c(B_2(G)) = 3$.

Theorem 3.5: Let G be any graph such that $B_2(G)$ is connected and $\gamma_c(B_2(G)) \geq 4$. Then $\gamma_c(B_2(G)) \leq 5$.

Proof:

Case(i): G contains triangles.

Let v_1, v_2, v_3 be the vertices of a triangle in G and e'_{12}, e'_{23} and e'_{13} be the line vertices in $B_2(G)$ corresponding to the edges (v_1, v_2), (v_2, v_3) and (v_1, v_3) respectively. Then, $\{v_1, e'_{12}, e'_{23}, e'_{13}\}$ is a connected dominating set of $B_2(G)$. Hence, $\gamma_c(B_2(G)) \leq 4$.

Case(ii): G is triangle free.

Since $B_2(G)$ is connected and $\gamma_c(B_2(G)) \geq 4$, G contains P_4, $P_3 \cup K_2$ or $3K_2$ as a subgraph. If G contains P_4 as a subgraph, then there exists a dominating set D of $B_2(G)$ such that <D> $\cong K_1+K_1+K_2$ in $B_2(G)$. If G contains $P_3 \cup K_2$ as a subgraph, then there exists a dominating set D of five vertices in $B_2(G)$ such that <D> $\cong P_5$ in $B_2(G)$. Similarly, if G contains $3K_2$ as a subgraph, then there exists a dominating set D of $B_2(G)$ with <D> $\cong P_4$ in $B_2(G)$. Hence, $\gamma_c(B_2(G)) \leq 5$.

The bound $\gamma_c(B_2(G)) \leq 4$ is attained, when $G \cong K_n$, $n \geq 5$ and P_m, $m \geq 7$.

Remark 3.5: From the above theorems, it follows that, the total domination number $\gamma_t(B_2(G))$ is also at most 4, since if G contains $P_3 \cup K_2$ as a subgraph, then there exists a total dominating set of $B_2(G)$ containing 4 vertices.

Similarly, the following propositions can be proved.

Proposition 3.4: Cycle domination number γ_0 of $B_2(G)$ is 3 if and only if either there exists a dominating set D of L(G) with <D> $\cong C_3$ in L(G). That is, $\gamma_0(L(G)) = 3$ or $\overline{G}$ contains $P_3 \cup K_1$ as an induced subgraph such that the edges of G incident with the vertex in K_1 are adjacent to at least on of the edges in P_3.

Proposition 3.5: $\gamma_0(B_2(G)) \leq \gamma_0(L(G))$.

In the following, the graphs G for which the perfect domination number γ_p of $B_2(G)$ is 2 or 3 are obtained.

Theorem 3.6.: For any graph G with at least two edges, $\gamma_p(B_2(G)) = 2$ if and only if $G \cong 2K_2$.

Proof: Assume $\gamma_p(B_2(G)) = 2$. Then there exists a perfect dominating set D of $B_2(G)$ containing two vertices. But, by Proposition 6.3.2., $\gamma(B_2G)) = 2$ if and only if $\gamma_i'(G) = 2$. Hence, D must contain two line vertices such that the corresponding edges, say e_1 and e_2 in G are independent. Since D is a perfect dominating set of $B_2(G)$, each line vertex (or point vertex) in $B_2(G) - D$ is adjacent to exactly one of the vertices in D. That is, there exists no edge in G adjacent to at least one of e_1 and e_2. Hence, $G \cong 2K_2$. Converse is obvious.

Theorem 3.7: For any graph G not totally disconnected, $\gamma_p(B_2(G)) = 3$ if and only if G is one of the graphs C_3, P_3 and $K_2 \cup mK_1$, $m \geq 1$.

Proof: Assume $\gamma_p(B_2(G)) = 3$. Then there exists a perfect dominating set D of $B_2(G)$ having three vertices. If all the vertices of D are point vertices, then $G \cong P_3$, C_3 or $K_2 \cup K_1$. If D contains two point vertices and one line vertex, then $G \cong K_2 \cup mK_1$, $m \geq 1$. If all the vertices of D are line vertices, then $G \cong C_3$. Converse is obvious.

Remark 3.6: There exists no perfect dominating set containing at least four vertices in $B_2(G)$.

In the following, the point-set domination number γ_{ps} of $B_2(G)$ is determined by applying Theorem 2.1. We give a lower bound for $\gamma_{ps}(B_2(G)$.

Theorem 3.8: $\gamma_{ps}(B_2(G)) \geq 3$.

Proof: By Theorem 3.1., $\gamma(B_2G)) = 2$ if and only if $\gamma_i'(G) = 2$. That is, any 2-dominating set of $B_2(G)$ contains only line vertices such that the corresponding edges in G are independent. Hence, any point-set dominating set (psd-set) D of $B_2(G)$ with $|D|= 2$ will contain only line vertices. Let $D = \{e'_1, e'_2\}$ be a dominating set of $B_2(G)$ and $e_1 = (v_1, v_2)$, $e_2 = (v_3, v_4)$ be the corresponding edges in G, where $v_1, v_2, v_3, v_4 \in V(G)$. Then, e_1 and e_2 are independent in G and for the independent set $W = \{v_1, v_3\}$ in $V(B_2(G))-D$, there exists no vertex in D adjacent to both v_1 and v_3 and hence D is not a psd-set of $B_2(G)$ by Theorem 3.1. Therefore, $\gamma_{ps}(B_2(G)) \geq 3$.

This bound is attained, when $G \cong K_{1,3}$.

Remark 3.7: If radius of L(G) is 1, then $\gamma_{ps}(B_2(G)) = 3$.

An upper bound for $\gamma_{ps}(B_2(G))$ is given in terms of number of edges in G, as follows.

Theorem 3.9: For any graph G having at least one edge, $\gamma_{ps}(B_2(G)) \leq q + 2$.

Proof: Let v_1 and v_2 be any two adjacent vertices in G. Then $D = V(L(G)) \cup \{v_1, v_2\}$ is a psd-set of $B_2(G)$ and hence $\gamma_{ps}(B_2(G)) \leq q + 2$.

This bound is attained, if $G \cong 2K_2 \cup K_1$.

Theorem 3.10: $\gamma_{ps}(B_2(G)) \leq (p/2) + 2$, if there exists a unique perfect matching in G.

Proof: Let M be the perfect matching in G and $e \in M$. Since each element in M is an edge in G, let $e = (u, v)$, where $u, v \in V(G)$. Then the line vertices in $B_2(G)$ corresponding to the edges in M together with the vertices u, v is a psd-set of $B_2(G)$. Hence, $\gamma_{ps}(B_2(G)) \leq (p/2)+2$.

Remark 3.8: (i). For any graph G having at least 3 vertices, the set of all point vertices in $B_2(G)$ is a psd-set of $B_2(G)$ if and only if there exists no perfect matching in G.

(ii). Any dominating set of $B_2(G)$ containing line vertices only is not a psd-set of $B_2(G)$.

In the following, the restrained domination number γ_r of $B_2(G)$ is obtained. We find the graphs G for which $\gamma_r(B_2(G))$ is 2.

Theorem 3.11: $\gamma_r(B_2(G)) = 2$ if and only if there exist two independent edges e_1 and e_2 in G such that each edge in G is adjacent to at least one of e_1 and e_2 and for every $v \in V(G)$, there exists an edge in $G-\{e_1, e_2\}$ not incident with v.

Proof: Assume $\gamma_r(B_2(G)) = 2$. Then there exists a restrained dominating set D of $B_2(G)$ containing two vertices. Since D is a 2-dominating set of $B_2(G)$, the vertices of D must be line vertices and the corresponding edges in G are independent and each edge in G is adjacent to at least one of the above independent edges. Let there exist a vertex v in G such that all the edges in $G-\{e_1, e_2\}$ are incident with v, then $v \in V(B_2(G))-D$ is not adjacent to any of the vertices in $V(B_2(G))-D$, which is a contradiction.
Conversely, assume there exist two independent edges e_1 and e_2 in G such that each edge in G is adjacent to at least one of e_1 and e_2 and for every $v \in V(G)$, there exists an edge in

$G-\{e_1, e_2\}$ not incident with v. Let D be the set of line vertices in $B_2(G)$ corresponding to the edges in e_1 and e_2. Let w be a point vertex in $V(B_2(G))-D$. By the assumption, w is adjacent to a line vertex in $V(B_2(G))-D$. Also each line vertex in $V(B_2(G))-D$ is adjacent to either a point vertex or a line vertex in $V(B_2(G))-D$, since each line vertex is adjacent to p–2 point vertices. Hence, D is a restrained dominating set of $B_2(G)$ and $\gamma_r(B_2(G)) \leq 2$. But $\gamma_r(B_2(G)) \geq 2$. Therefore, $\gamma_r(B_2(G)) = 2$.]

The next theorem relates $\gamma_r(B_2(G))$ with the line independence number α_1 of G.

Theorem 3.12: If $G \neq C_3, K_{1,n}, n \geq 1$, then $\gamma_r(B_2(G)) \leq \alpha_1(G) + 1$.

Proof: Let D be a line cover of G with $|D| = \alpha_1(G)$. Since $G \neq C_3, K_{1,n}, n \geq 1$, D contains at least two independent edges. Let D' be the line vertices in $B_2(G)$ corresponding to the edges in D. If $<E(G)-D> \neq K_{1,n} \cup mK_1$, for $n \geq 1$ and $m \geq 0$, then D' is a restrained dominating set of $B_2(G)$ and $\gamma_r(B_2(G)) \leq \alpha_1(G)$. If $<E(G)-D> \cong K_{1,n} \cup mK_1$, for $n \geq 1$ and $m \geq 0$, then $D' \cup \{$center vertex of $K_{1,n}\}$ is a restrained dominating set of $B_2(G)$ and in this case, $\gamma_r(B_2(G)) \leq \alpha_1(G) + 1$.

Remark 3.9: $\gamma_r(B_2(G)) = \alpha_1(G)$, if $G \cong C_4$ and $\gamma_r(B_2(G)) = \alpha_1(G)+1$, if $G \cong K_1+K_1+K_2$.

Similarly, a relationship between $\gamma_r(B_2(G))$ and the point covering number α_0 of G can be given.

Theorem 3.13: $\gamma_r(B_2(G)) \leq \alpha_0(G) + 1$, if there exists a point cover D of G with $|D| = \alpha_0(G) \geq 3$ such that $<D>$ is not independent.

Remark 3.10:

(i). For any connected graph G with at least 3 vertices, the set of all point vertices is a restrained dominating set of $B_2(G)$.

(ii). Since no two point vertices in $B_2(G)$ are adjacent, the set of all line vertices is not a restrained dominating set of $B_2(G)$.

In the following, the split domination number γ_s of $B_2(G)$ is determined. Here, the graphs G for which both G and $B_2(G)$ are connected, are considered.

Theorem 3.14: Let $\beta_1(G) \geq 2$ and e_1, e_2 be two independent edges in G. Then the set of line vertices in $B_2(G)$ corresponding to the edges e_1, e_2 is a split dominating set of $B_2(G)$ if and only if $G-\{e_1, e_2\} \cong 2K_2$ or $K_{1,n} \cup mK_1$, for $n \geq 1$ and $m \geq 0$.

Proof: Let e'_1, e'_2 be the line vertices in $B_2(G)$ corresponding to the independent edges e_1 and e_2 and $D = \{e'_1, e'_2\}$. Since D is a dominating set of $B_2(G)$, each edge in G is adjacent to at least one of the edges e_1 and e_2. If $G-\{e_1, e_2\} \neq 2K_2$ and $K_{1,n} \cup mK_1$, for $n \geq 1$ and $m \geq 0$, then $\langle V(B_2(G))-\{e'_1, e'_2\}\rangle$ is connected, which is a contradiction. Converse follows easily.

In the following, an upper bound for $\gamma_s(B_2(G))$ is given.

Theorem 3.15: $\gamma_s(B_2(G)) \leq q - \Delta(G) + 2$, where $\Delta(G)$ is the maximum degree of G.

Proof: Let v be a vertex of maximum degree in G. Since $B_2(G)$ is connected, $G \neq K_{1,n}$, for

$n \geq 1$ and hence there exists at least one edge $e = (u, w)$, where $u, w \in V(G)$ in G not incident with v. Let D' be the set of line vertices in $B_2(G)$ corresponding to the edges in G not incident with v. Then $D = \{u, w\} \cup D'$ is a dominating set of $B_2(G)$ and v is an isolated vertex in $V(B_2(G))-D$ and hence $\langle V(B_2(G))-D\rangle$ is disconnected . Thus, D is a split dominating set of $B_2(G)$. Therefore, $\gamma_s(B_2(G)) \leq |D| = q - \Delta(G) + 2$.

This bound is attained, if $G \cong C_3$.

Theorem 3.16: Let $\delta(G) \geq 2$ and v be a vertex of maximum degree in G. If the subgraph of G induced by $V(G)-\{v\}$ contains at least two independent edges, then $\gamma_s(B_2(G)) \leq q - \Delta(G)$.

Proof: Let D be the set of line vertices in $B_2(G)$ corresponding to the edges not incident with v in G. Since $\langle V(G)-\{v\}\rangle$ contains at least two independent edges, each point vertex in $V(B_2(G))$–D is adjacent to at least one vertex in D. Since $\delta(G) \geq 2$, D dominates each line vertex in $V(B_2(G))$–D and v is isolated in $<V(B_2(G))–D>$. Hence, D is a split dominating set of $B_2(G)$ and $\gamma_s(B_2(G)) \leq |D| = q - \Delta(G)$.

This bound is attained, if $G \cong C_5$.

Theorem 3.17: If $\beta_1(G) = 2$, then $\gamma_s(B_2(G)) \leq p + q - 6$, where $\beta_1(G)$ is the line independence number of G.

Proof: Since $\beta_1(G) = 2$, G contains $2K_2$ as an induced subgraph. Let $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ be the edges of $2K_2$, where $u_1, v_1, u_2, v_2 \in V(2K_2)$. Then $D' = \{u_1, v_1, u_2, v_2, e'_1, e'_2\} \subseteq V(B_2(G))$, where e'_1 and e'_2 are the line vertices corresponding to the edges e_1 and e_2 respectively. If $D = V(B_2(G)) - D'$, then D is a dominating set of $B_2(G)$. Also $<V(B_2(G)) - D> \cong 2K_{1,3}$ in $B_2(G)$ and is disconnected. Hence, D is a split dominating set of $B_2(G)$. Thus, $\gamma_s(B_2(G)) \leq p + q - 6$.

This bound is attained if $G \cong C_4$.

Remark 3.11:

(i). The set of all point (or line) vertices in $B_2(G)$ is a split domination set $B_2(G)$, if G (or L(G)) is disconnected

(ii). If $G \cong P_3 \cup K_2$, then there exists a split domination set of $B_2(G)$ having two vertices.

(iii). If $G \cong P_3 \cup mK_2$, for $m \geq 2$, then there exists a split domination set of $B_2(G)$ having three vertices.

Example 3.1:

(i). $\gamma_s(B_2(P_n))$ = 2, if n = 4; and
= n-3, if $n \geq 5$.

(ii). $\gamma_s(B_2(C_n))$ = 2, if n = 4; and
= 3, if n = 3 and $n \geq 5$.

(iii). $\gamma_s(B_2(K_n))$ = ((n - 1)(n - 2)/2), if $n \geq 4$.

In the following, the upper bounds of non-split domination number γ_{ns} of $B_2(G)$ are obtained.

Theorem 3.18: For any graph G having at least six vertices and two adjacent edges, $\gamma_{ns}(B_2(G)) \leq 4$.

Proof: Let $P_3 : v_1\, v_2\, v_3$ be a path of length 2 in G and $e_{12} = (v_1, v_2)$, $e_{23} =$

$(v_2, v_3) \in E(G)$. Let e'_{12}, e'_{23} be the corresponding line vertices in $B_2(G)$ and let $D = \{v_2, e'_{12}, e'_{23}\}$. Then, $D \subseteq V(B_2(G))$. If $G-\{e_{12}, e_{23}\} \neq K_{1,n} \cup mK_1$, for $n \geq 1$ and $m \geq 0$, then D is a non-split dominating set of $B_2(G)$. Hence, $\gamma_{ns}(B_2(G)) \leq 3$. If $G-\{e_{12}, e_{23}\} \cong K_{1,n} \cup mK_1$, for $n \geq 1$ and $m \geq 0$, then $D \cup \{$center vertex of $K_{1,n}\}$ is a non-split dominating set of $B_2(G)$. Hence, $\gamma_{ns}(B_2(G)) \leq 4$.

This bound is attained, if $G \cong 2P_3$.

Theorem 3.19: If $\beta_1(G) \geq 3$, then $\gamma_{ns}(B_2(G)) \leq p + q - 3\beta_1(G)$.

Proof: Let $\beta_1(G) = n$, where $n \geq 3$ and $e_1, e_2, \ldots, e_n$ be the n independent edges in G, where $e_i = (u_i, v_i) \in E(G)$, $u_i, v_i \in V(G)$, $i = 1,2,\ldots,n$. Let $e_i$be the line vertices in $B_2(G)$ corresponding to the edges e'_i, $(i = 1,2,\ldots,n)$. Then $D' = \{u_1, v_1, \ldots, u_n, v_n, e'_1, e'_2, \ldots, e'_n\} \subseteq V(B_2(G))$ and $D = V(B_2(G)) - D'$ is a non-split dominating set of $B_2(G)$. In fact, $\langle V(B_2(G) - D\rangle$ is a bipartite subgraph of $B_2(G)$. Hence, $\gamma_{ns}(B_2(G)) \leq p + q - 3\beta_1(G)$.

This bound is attained, if $G \cong P_6$.

Example 3.2:

(i). $\gamma_{ns}(B_2(P_n))$ = 4, if n = 4;
= 2, if n = 6; and
= 3, if n = 5 and $n \geq 7$.

(ii). $\gamma_{ns}(B_2(C_n))$ = 4, if n = 4;
= 2, if n = 5, 6;
= 3, if n = 3 and $n \geq 7$.

(iii). $\gamma_{ns}(B_2(K_n))$ = 2, if n = 4, 5; and
= 3, if $n \geq 6$.

(iv). $\gamma_{ns}(B_2(K_{n,n}))$ = 4, if n = 2; and
= 3, if $n \geq 3$.

References

[1] J. Akiyama, T. Hamada and I. Yoshimura, Miscellaneous properties of Middle graphs, Tru. Math., Vol. 10 (1974), 41-52.

[2] M. Behzad and G. Chartrand, Total graphs and Traversability, Proc. Edinburgh Math. Soc., 15 (1966), 117-120.

[3] S. B. Chikkodimath and E. Sampathkumar, Semi total graphs-II, " Graph Theory Research Report", Karnatak University, No. 2, (1973), 5-9.

[4] E.J. Cockayne, R.M. Dawes and S.T. Hedetniemi, Total domination in graphs, Networks, 10 (1980), 211-219.

[5] E.J. Cockayne, B.L. Hartnell, S.T. Hedetniemi and R. Laskar, Perfect domination in graphs, J. Comb. Inf. Syst. Sci., 18 (1993), 136-148.

[6] G. S. Domke, J. H. Hattingh, S.T. Hedetniemi, R. Laskar and L.R. Markus, Restrained domination in graph, Discrete Math., 203 (1999), 61-69.

[7] T. Hamada and I. Yoshimura, Traversability and connectivity of the Middle graph of a graph, Discre. Math., 14 (1976), 247-256.

[8] F. Harary, Graph Theory, Addison- Wesley, Reading Mass, (1972).

[9] T. N. Janakiraman, S. Muthammai, M. Bhanumathi, On the Boolean function graph of a graph and on its complement, Math. Bohem., 130 (2005), No. 2, 113 – 134.

[10] T. N. Janakiraman, S. Muthammai, M. Bhanumathi, Domination numbers on the Boolean function graph of a graph, Math. Bohem., 130 (2005), No. 2, 135 – 151.

[11] T. N. Janakiraman, S. Muthammai, M. Bhanumathi, Global Domination and Neighborhood numbers in Boolean function graph of a graph, Math. Bohem., 130 (2005), No.3, 231 – 246.

[12] T. N. Janakiraman, S. Muthammai, M. Bhanumathi, Domination numbers on the complement of the Boolean function graph of a graph, Math. Bohem., 130 (2005), No. 3, 247 – 263.

[13] T. N. Janakiraman, S. Muthammai, M. Bhanumathi, On the Boolean function graph B($\overline{K_p}$, NINC, L(G)) of a graph, Proceedings of the National Conference on Mathematical Techniques and Applications (NCMTA 2007)Jan 5 & 6, 2007, S.R.M. University, Chennai. Pp.267-276. © 2008 Narosa Publishing House, New Delhi.

[14] V.R. Kulli and B. Janakiram, The non-split domination number of a graph, IndianJ.pure appl. Math., 27(6): 537-542, June 1996.

[15] O. Ore, Theory of graphs, Amer. Math. Soc. Colloq. Publ., 38, Providence(1962).

[16] E. Sampathkumar and L. Pushpalatha, Point-set domination number of a graph, Indian J. pure appl. Math., 24 (4): (1993), 225-229.

[17] E. Sampathkumar and H.B. Walikar, The connected domination number of a graph, J. Math.Phys. Sci., 13 (6), 607-613.

[18] D.V.S. Sastry and B. Syam Prasad Raju, Graph equations for Line graphs, Total graphs, Middle graphs and Quasi-total graphs, Discrete Mathematics, 48(1984), 113 - 119.

[19] H. Whitney, Congruent graphs and the connectivity of graphs, Amer. J.Math. 54 (1932), 150- 168.