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K- Equitable Labeling of Graphs

N. Sridharan[1] and R. Umarani[2]

[1] *H.O.D. of Mathematics, Alagappa University, Karaikudi*

[2] *Director of Collegiate Education, Chennai*

Abstract: *Cahit introduced the k- equitable labeling as a generalization of graceful labeling. In this paper, we study on k- equitable labeling and we prove that the graph* P_n^+ *is k- equitable for all k, n and the graph* C_n^+ *is k-equitable if* $n \neq \frac{k}{2} (\bmod k)$ *or* $k \neq 2n+1$.

1. Introduction

Cahit has introduced a variation of both graceful and harmonious labelings[1 – 2]. Let f be a function from the vertices of G to {0,1} and for each edge xy assign the label $|f(x) - f(y)|$ and call f a cordial labeling of G if the number of vertices labeled 0 and the number of vertices labeled 1 differ at most by 1 and the number of edges labeled 0 and the number of edges labeled 1 differ at most by 1.

In 1990, Cahit proposed the idea of distributing the vertex and edge labels among {0,1,……..k-1}as evenly as possible to obtain a generalization of graceful labeling as follows: For a graph G (V,E) and a positive integer k, assign vertex labels from {0,1, ……,k-1} so that when the edge labels introduced by the absolute value of the difference of the vertex labels, the number of vertices labeled with i and the number of vertices labeled with j differ by at most one and the number of edges labeled with i and the number of edges labeled with j differ by atmost one. Cahit has called a graph with such an assignment of labels k-equitable.

A graph G (V, E) is graceful if and only if it is $|E(G)| + 1$ – equitable. G is cordial if and only if it is 2-equitable. Szaniszlo has proved that P_n is k-equitable for all k[4]. In this paper, we study on k- equitable labeling and we prove that the graph P_n^+ is k- equitable for all k,n. For an extensive survey on graph labeling we refer to Gallian[3].

2. Main Result

Theorem 2.1 : If P_n is the path on *n* vertices, the graph P_n^+ is k-equitable for any k, $n \in N$.

Proof : Let Pn be the path $v_1v_2\ldots.v_n$ and let $v'_1,v'_2\ldots.v'_n$ be the pendant vertices adjacent to $v_1,v_2,\ldots,v_n$ respectively in P_n^+. Let k be a given positive integer.

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Let n = mk + t where $1 \leq t \leq k$.

Define a map f: $V(P_n^+) \longrightarrow \{0,1,2,\ldots.k-1\}$ as

a) If k is even,

$$\left.\begin{array}{l} f(v'_{2i+1}) = 2i \\ f(v_{2i+1}) = k-2i-1 \end{array}\right\} \text{ for } i = 0,1,2,\ldots, \frac{k}{2}-1$$

$$\left.\begin{array}{l} f(v'_{2i}) = k-2i \\ f(v_{2i}) = 2i-1 \end{array}\right\} \text{ for } i = 0,1,2,\ldots, \frac{k}{2}$$

$f(v_{kl+i}) = f(v_i)$; $f(v'_{kl+i}) = f(v'_i)$ for all l= 1,2,.....,m, i = 1,2,....k if $l \neq m$
i = 1, 2....,t if l =m

b) If k is odd,

$$\left.\begin{array}{l} f(v'_{2i+1}) = 2i \\ f(v_{2i+1}) = k-2i-1 \end{array}\right\} \text{ for } i = 0,1,2,\ldots, \frac{k-1}{2}$$

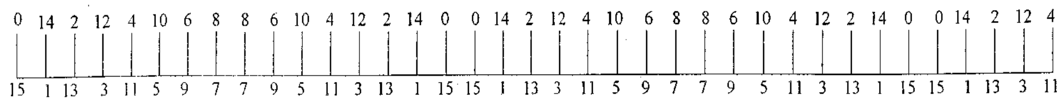
$$\left.\begin{array}{l} f(v'_{2i}) = k-2i \\ f(v_{2i}) = 2i-1 \end{array}\right\} \text{ for } i = ,1,2,\ldots, \frac{k-1}{2}$$

$$\left.\begin{array}{l} f(v'_{k+j}) = f(v_{k-j+1}) \\ f(v_{k+j}) = f(v_{k-j+1}) \end{array}\right\} \text{ for } j = 1,2,\ldots,k$$

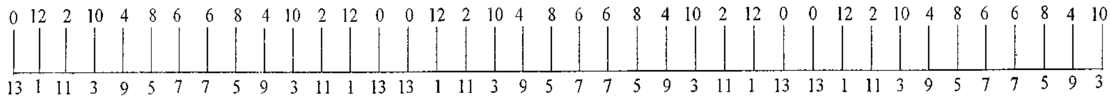
$$\left.\begin{array}{l} f(v'_{2kl+i}) = f(v_i) \\ f(v_{2kl+i}) = f(v_i) \end{array}\right\} \text{ for } l = 1,2\ldots.., \left[\frac{n}{2k}\right] \text{; and}$$

for i = 1,2,....2k. if $l \neq \left[\frac{n}{2k}\right]$ and i = 1,2,.... (n-$\left[\frac{n}{2k}\right]$ 2k) if $l = \left[\frac{n}{2k}\right]$

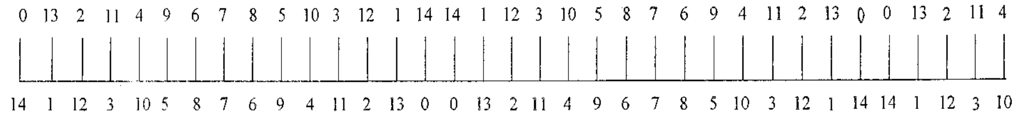
One can verify that the number of vertices labelled with i $V_f(i)$ and the number of vertices labelled with j $V_f(j)$ and the number of edges labelled with i $e_f(i)$ and the number of edges labelled with j differs by at most one for all I and j. Thus the graph P_n^+ is K-equitable for all n and k. K- equitable labeling for P_{37}^+, P_{39}^+ and P_{35}^+ are shown in Figure2.1.



An 16-Equitable labeling for P_{37}^{+}



An 14-Equitable labeling for P_{39}^{+}



An 15-Equitable labeling for P_{35}^{+}

Fig 2.1

Theorem 2.2 : Let n and k be positive integers such that $n \neq \frac{k}{2}$ (mod k) or $k \neq 2n+1$, then C_n^+ is k- equitable.

Proof: Let G = (V,E) be the graph C_n^+. Let $v_1, v_2, \ldots\ldots v_n$ be the cycle C_n in G and let for each $i(1 \le i \le n)$, v_i'be the vertex of degree one and adjacent to the vertex v_i.

First we assume that $k < n$.

Let $n = mk + r$, where $0 \le r < k$ Define $f : V(G) \longrightarrow \{0,1,2,\ldots,k-1\}$ as follows:

$$f(v_{2kl+j}) = \begin{cases} i-1 \text{ if } i \text{ is odd and } 1 \le i \le k; l = 0,1,\ldots,\left[\frac{m}{2}\right] \\ k-1 \text{ if } i \text{ is even and } 1 \le i \le k; l = 0,1,\ldots,\left[\frac{m}{2}\right] \end{cases}$$

$$f\left(v_{(2l+1)k+i}\right) = f(v_{k-i+1}) \text{ for } l = 0,1,\ldots, \left[\frac{m}{2}\right] \text{ and } i = 1,\ldots,k.$$

$$f(v'_i) = (k-1) - f(v_i) \text{ for all } 1 \le i \le n.$$

We consider various cases and in each case we define a k- equitable labeling $g: V(G) \longrightarrow \{0,1,\ldots,k-1\}$ by modifying the function f.

Case (i) Let k be even.

Subcase (i) Let r =1

Take $g\left(v_{mk+1}\right) = k-1$; $g(v'_1) = 0$; $g\left(v'_{mk+1}\right) = 1$ and

$g(v_i) = f(v_i)$ for all $i \le mk$. $g(v'_i) = f(v'_i)$ for all $1 < i \le mk$

Thus g is a k-equitable labeling for C^{+}_{mk+1} (k is even).

(An **14 - equitable labeling for** C^{+}_{29} **is** is illustrated in the figure 2.2).

Subcase (ii) r is odd $1 < r \le \frac{k+1}{3}$:

If $r \le \frac{k+1}{3}$ we have r-1 ≤ k-2r In this case we take g = f, without any modification.

(ie, We define g as g(u) = f(u) for all u $\in V(G)$)

(An **16 - equitable labeling for** C^{+}_{37} is shown in figure 2.3)

Subcase(iii) r is odd, $\frac{k-1}{3} < r < \frac{k}{2}$ and $\frac{k-r+1}{2}$ is odd

$$g(v_{mk+j}) = \begin{cases} j & \text{if } j = \frac{k-r+1}{2}, \frac{k-r+1}{2}+2, \ldots, r-2 \\ r+1 & \text{if } j = r \end{cases}$$

$$g(v'_{mk+j}) = \left\{ j \quad \text{if } j = \frac{k-r+1}{2}+1, \frac{k-r+1}{2}+3, \ldots, r-1 \right.$$

and $g(v_i) = f(v_i)$ if $i \ne mk+j$ where $j \in \left\{\frac{k-r+1}{2}, \frac{k-r+1}{2}+2, \ldots, r-2, r\right\}$

$g(v'_i) = f(v'_i)$ if $i \ne mk+j$ where $j \in \left\{\frac{k-r+1}{2}+1, \frac{k-r+1}{2}+3, \ldots, r-1\right\}$

(A **24 – equitable labeling for** C^{+}_{35} is shown in the Figure 2.4)

Subcase(iv) r is odd $\frac{k-1}{3} < r < \frac{k}{2}$ and $\frac{k-r+1}{2}$ is even i.e., (r-1 = k mod 4)

Let $g(v'_{mk+j}) = j$ if j is even and $j \ge \frac{k-r+1}{2}$

$g(v_{mk+j}) = j$ if j is odd and $j \ge \frac{k-r+1}{2}$

and $g(v_i) = f(v_i)$ for all other v_i and $v'i$ $g(v'_i) = f(v'_i)$ for all other vertices

(A **26 – equitable labeling for** C^{+}_{37} is shown in the Figure 2.5).

Subcase (v) $\frac{k}{2} < r < k$ and r is odd

In this case we take g = f .

(In the figure 2.6 , a **26 equitable labeling for** C^{+}_{45} is shown)

Now we consider the cases when r is even.

Subcase (vi) Let r = 2

(As $r \neq \frac{k}{2}$ and k is even we have k =2 or k ≥ 6 but k ≠ 2 , as $r < k$).

Assume that k≥ 6

We define g as

$g(v_{mk+1}) = 0;$ $g(v_{mk+2}) = k-1$

$g(v'_{mk+1}) = 2;$ $g(v'_{mk+2}) = 1$ and $g(v'_1) = 3$

and g(u) = f(u) for all other vertices u.

(A **18 - equitable labeling for** C_{38}^{+} is shown in the figure 2.7)

Subcase (vii) r is even and $2 < r \leq \frac{k+2}{3}$.

Let $g(v_{mk+r}) = r$ and g(u) = f(u) for all $u \in V(G) - \{v_n\}$

(See the figure 2.8 ,for a **18- equitable labeling for** C_{40}^{+} .)

Subcase (viii) r is even and $\frac{k+2}{3} < r < \frac{k}{2}$

If r = 0 (mod 4) we define g as follows:

$g(v_{mk+j}) = j$ for all $j = \frac{r}{2}+1, \frac{r}{2}+3, \ldots, r$

$g(v'_{mk+j}) = j$ for all $j = \frac{r}{2}+2, \frac{r}{2}+4, \ldots, r-1$

g(u) = f(u) for all other $u \in V(G)$

if r =2 (mod 4) we define g as

$g(v_{mk+j}) = j$ for all $j = \frac{r}{2}+1, \frac{r}{2}+3, \ldots, r$

$g(v'_{mk+j}) = j$ for all $j = \frac{r}{2}+2, \frac{r}{2}+4, \ldots, r-1$

$g(v_n) = k-r-1$ and g(u) = f(u) for all other vertices $u \in V(G)$

(In the figure 2.9, a **28 - equitable labeling for** C_{40}^{+} is shown.)

Subcase(ix) r is even and $\frac{k}{2} < r < \frac{2k+1}{3}$

In this case we make no changes in f and we take g = f

(In the figure 2.10 a **28 - equitable labeling for** C_{44}^{+}))

Subcase(x) r is even and $\frac{2k+1}{3} < r < k$

If r = 0 (mod 4) define

$g(v_{mk+j}) = j$ for all $j = \frac{r}{2}+1, \frac{r}{2}+3, \ldots, r-1$

$g\left(v'_{mk+j}\right) = j$ for all $j = \frac{r}{2}+2, \frac{r}{2}+4, \ldots, r$. $g(u) = f(u)$ for all other vertices u

if r =2 (mod 4) define

$g\left(v_{mk+j}\right) = j$ for all $j = \frac{r}{2}+2, \frac{r}{2}+4, \ldots, r-1$

$g\left(v'_{mk+j}\right) = j$ for all $j = \frac{r}{2}+1, \frac{r}{2}+3, \ldots, r$; $g\left(v_n\right) = k-r-1$

and g(u) = f(u) for all other vertices $u \in V(G)$.

Case (ii) Let k be odd.

Subcase (i) Let r = 1 i.e., n = mk+1

If m is odd define g as follows: $g\left(v'_n\right) = 1$, and g(u) = f(u) for all $u \neq v'n \in V\left(c_n^+\right)$.

if m is even, then define the map g as follows: $g\left(v_n\right) = 0$, $g\left(v'_n\right) = k-1$; g(u) = f(u) for all $u \neq$, v_n, v'_n

Subcase (ii) r is odd and $1 < r < \frac{k+2}{3}$.

If m is odd, then define $g\left(v_n\right) = r-1$; $g\left(v'_n\right) = k-r$ and g(u) = f(u) for all other u .

If m is even we define g(u) = f(u) for all $u \in V\left(C_n^+\right)$

Subcase (iii) Let r = 2

If m is odd, we define g as follows: $g\left(v_{n-1}\right) = k-1$; $g\left(v_n\right) = 1$

$g\left(v'_{n-1}\right) = 0$; $g(P_n^+) = k-2$ and g (u) = f(u) for all $u \in V(G)$.

If m is even, we define g as follows:

$g\left(v_n\right) = (k-2)$; $g\left(v_{n-1}\right) = k-1$, $g\left(v'_n\right) = 1$; $g\left(v'_{n-1}\right) = k-1$; $g\left(v'_{n-2}\right) = 0$

and g (u) = f(u) for all $u \in V(G)$.

Subcase (iv) Let r be even and $2 < r < \frac{k+2}{3}$.

If m is odd, take g = f , i.e., g(u) = f(u) for all $u \in V(G)$.

If m is even, we define g as follows: $g\left(v_n\right) = r-1$; $g\left(v'_n\right) = k-r$ and

g(u) = f(u) for all other $u \in V\left(C_n^+\right)$

Subcase (v) r is odd, $\frac{k+2}{3} \leq r < \frac{k}{2}$.

Let m be odd. If r-1 = 0 (mod 4) We define g as follows:

$g\left(v'_{mk+j}\right) = j$ for $j = \frac{r+1}{2}, \frac{r+1}{2}+2, \ldots, r$ and $g\left(v_{mk+j}\right) = j$ for $j = \frac{r+1}{2}+1, \frac{r+1}{2}+3, \ldots, r-1$. and g(u) = f(u) for all other $u \in V(G)$.

If r-1 = 2 (mod 4) we define g as follows:

$g\left(v_{mk+j}\right)$ = k-j-1 for j = $\frac{r+3}{2}$, $\frac{r+3}{2}$+2 ,…..,r.

$g\left(v'_{mk+j}\right)$ = k-j-1 for j = $\frac{r+3}{2}$+1 , $\frac{r+3}{2}$+3,…..,r-1 and g(u) = f(u) for all other u.

Let m be even, r odd and $\frac{k-r+2}{2}$ be odd.

Define g as follows:

$g\left(v'_{mk+j}\right)$ = k-j-1 , for all odd j ≥ $\frac{k-r+2}{2}$.

$g\left(v_{mk+j}\right)$ = k-j-1, for all even j ≥ $\frac{k-r+2}{2}$ and g(u)=f(u) for all other u.

If m is even ,r odd and $\frac{k-r+2}{2}$ is even ,define as follows:

$g\left(v'_{mk+j}\right)$ = k-j-1 for j = $\frac{k-r}{2}$, $\frac{k-r}{2}$+2 ,…..,r.

$g\left(v_{mk+j}\right)$ = k-j-1 for j = $\frac{k-r}{2}$+1 , $\frac{k-r}{2}$+3 ,…..,r-1.

$g\left(v_{mk+j}\right)$ = j-2 for j = $\frac{k-r}{2}$.

$g\left(v'_{mk+j}\right)$ = j for j = $\frac{k-r}{2}$ -1.and g(u) = f(u) for all other u $\in v\left(C_n^+\right)$.

Subcase (vi): Let $\frac{k}{2} < r \le$ k-1.

We define the map g as follows: If both m and r are odd, let

$$g(v_{mk+j}) = \begin{cases} j-1 \text{ for all odd } j \ge \frac{k+1}{2}+1 \\ k-j \text{ for all even } j \ge \frac{k+1}{2}+1 \end{cases}$$

$$g(v'_{mk+j}) = \begin{cases} k-j \text{ for all odd } j \ge \frac{k+1}{2}+1 \\ j-1 \text{ for all even } j \ge \frac{k+1}{2}+1 \end{cases}$$

and g(u) = f(u) for all other vertices u .

(a) If both m and r are even, let

$$g(v_{mk+j}) = \begin{cases} j-1 \text{ for all even } j \geq \dfrac{k+1}{2}+1 \\ k-j \text{ for all odd } j \geq \dfrac{k+1}{2}+1 \end{cases}$$

$$g(v'_{mk+j}) = \begin{cases} k-j \text{ for all even } j \geq \dfrac{k+1}{2}+1 \\ j-1 \text{ for all odd } j \geq \dfrac{k+1}{2}+1 \end{cases}$$

and g(u) = f(u) for all other vertices u.

(b) If one of m and r is even and the other is odd, let

g(u) = f(u) for all $u \in V(G)$.

In all the above cases one can verify that $V_f(i)$ and $e_f(i)$ differs by at most one for all i. Thus the graph C_n^+ is K-equitable for all $n \neq \frac{k}{2}$ (mod k) or k≠ 2n+1, .

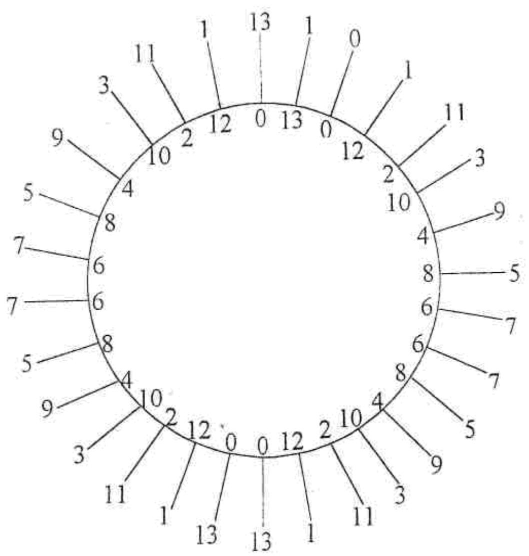


Fig 2.2 : A 14-equitable labelling for C_{29}^+

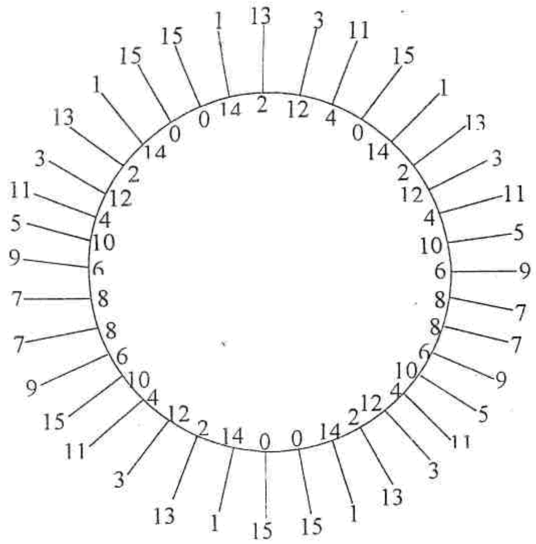


Fig 2.3 : A 16-equitable labelling for C_{37}^+

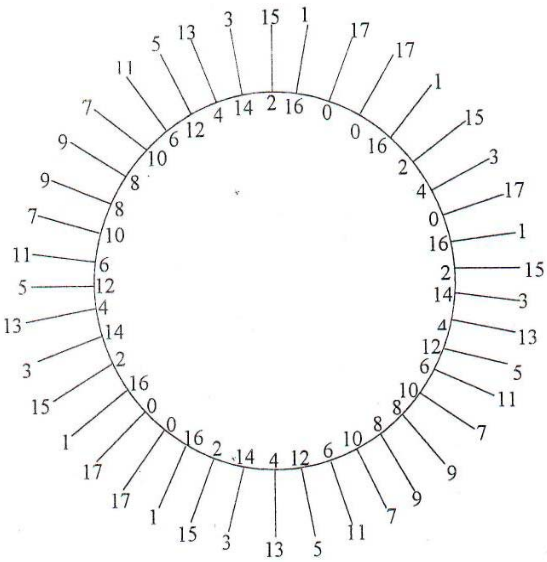


Fig 2.4 : A 24-equitable labelling for C_{35}^{+}

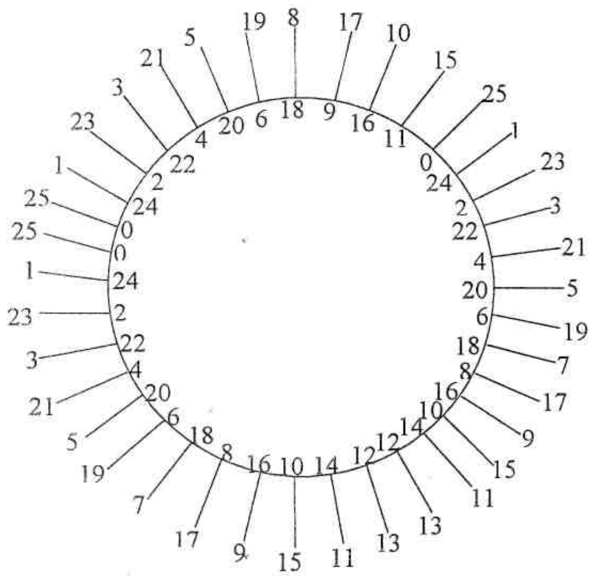


Fig 2.5 : A 26-equitable labelling for C_{37}^{+}

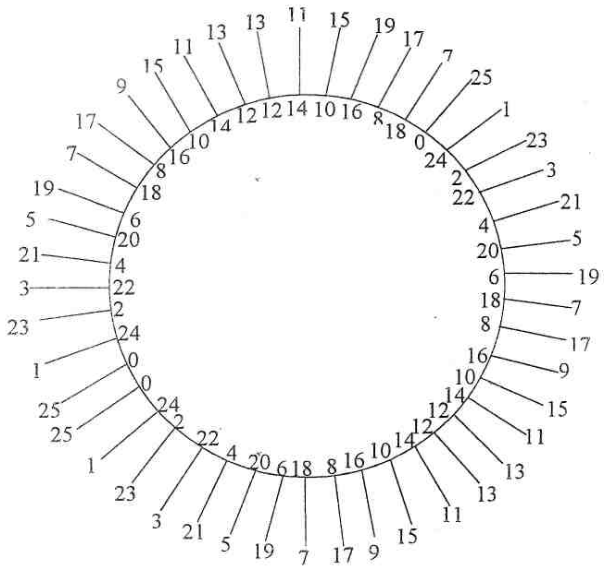


Fig 2.6 : A 26-equitable labelling for C_{45}^{+}

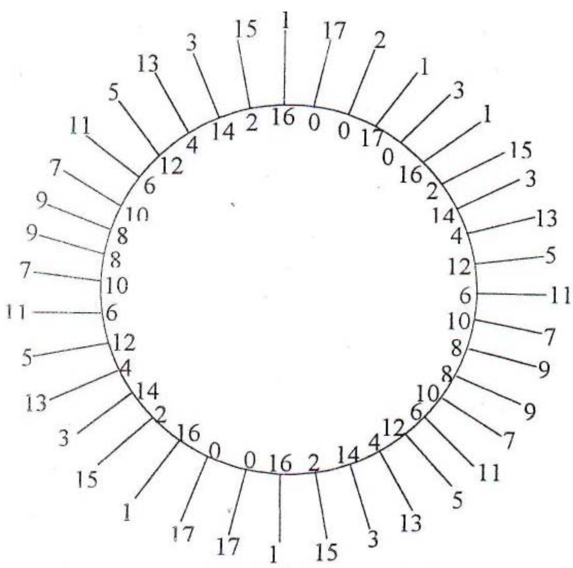


Fig 2.7 : An 18-equitable labelling for C_{38}^{+}

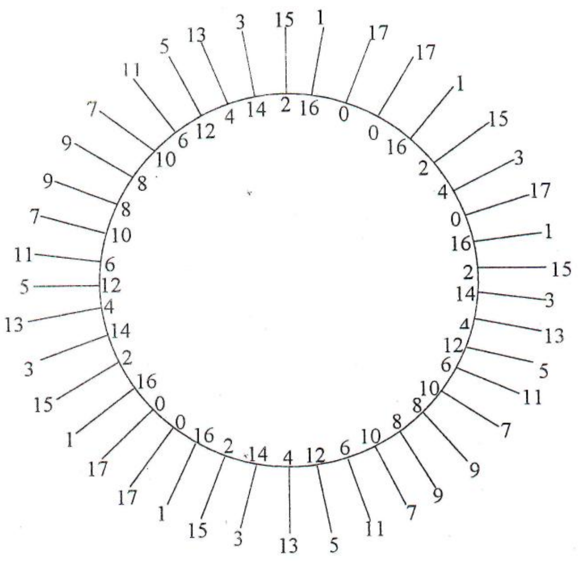


Fig 2.8 : A 18-equitable labelling for C^+_{40}

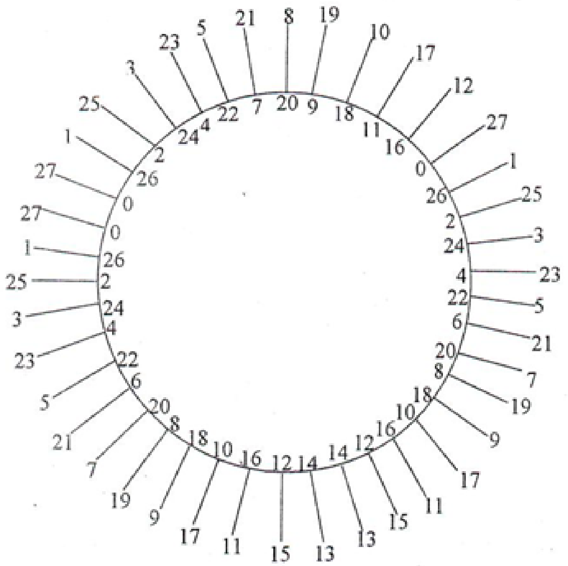


Fig 2.9 : A 28-equitable labelling for C^+_{40}

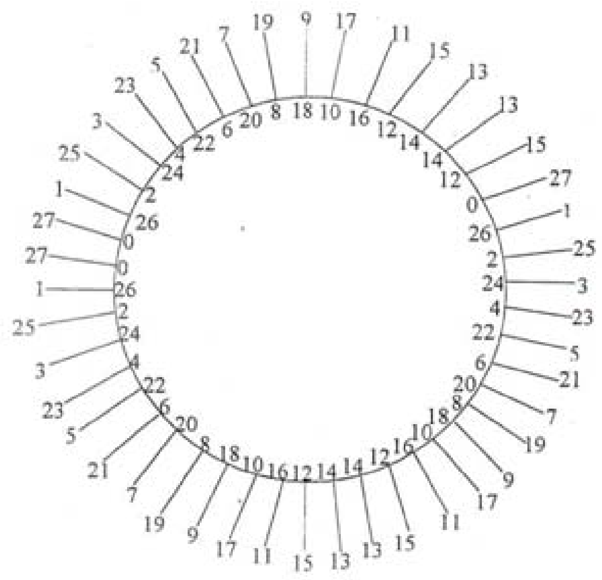


Fig 2.10 : A 28-equitable labelling for C^+_{44}

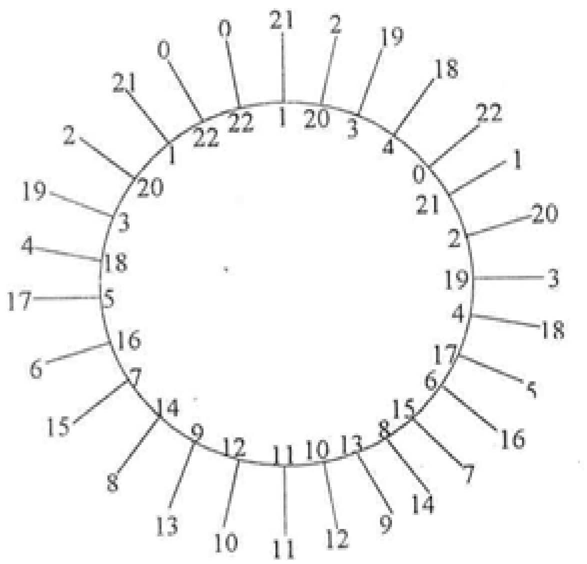


Fig 2.11 : A 23-equitable labelling for C^+_{28}

References

[1] I. Cahit, On Cordial and 3 – Equitable Labelings of Graphs, Utilita Mathematica, 37(1990), 189 - 198.

[2] I. Cahit, Equitable Tree Labelings, Ars Combinatoria, 140(1995), 279 – 286

[3] J.A. Gallian – A Dynamic Survey of Graph Labeling, Electronic J. Combinatorics, 5(1998) # DS6.

[4] Z.Szaniszlo, K-equitable labelings of cycles and some other graphs, Ars Combinatoria, 37(1994), 49 – 63.

[5] R.Umarani, A Study on graph labelings – k-Equitable and strong α' – labelings, Ph.D. Thesis, 2003.