International Journal of Engineering Sciences, Advanced Computing and Bio-Technology
Vol. 1, No. 4, October –December 2010, pp.183 -190

Even Distance Closed Domination in Graph

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Abstract: *In a graph G=(V,E), a set $S \subset V(G)$ is a distance closed set of G if for each vertex $u \in S$ and for each $w \in V$-S, there exists at least one vertex $v \in S$ such that $d_{<S>}(u, v) = d_G(u, w)$. Also, a vertex subset D of V(G) is a dominating set of G if every vertex in V-D is adjacent to at least one vertex in D. Combining the above concepts, a distance closed dominating set of a graph G is defined as follows: A subset $S \subseteq V(G)$ is said to be a distance closed dominating (D.C.D) set, if <S> is distance closed and S is a dominating set. In this paper, we define a new concept of domination called even distance closed domination (E.D.C.D) and we find various bounds for these parameters and characterized the graphs, which attain these bounds.*

Keywords: *domination number, distance, eccentricity, radius, diameter, self centered graph, neighborhood, induced sub graph, distance closed dominating set, even distance closed dominating set.*

1. Introduction

Graphs discussed in this paper are connected and simple only. For a graph, let V(G) and E(G) denotes its vertex and edge set respectively. The *degree* of a vertex v in a graph G is denoted by $deg_G(v)$. The length of any shortest path between any two vertices u and v of a connected graph G is called the *distance between u and v* and it is denoted by $d_G(u, v)$. The distance between two vertices in different components of a disconnected graph is defined to be ∞. For a connected graph G, the *eccentricity* $e_G(v) = \max \{d_G(u,v): \forall u \in V(G)\}$. If there is no confusion, we simply use the notion deg (v), d(u, v) and e(v) to denote degree, distance and eccentricity respectively for the connected graph. The minimum and maximum eccentricities are the *radius* and *diameter* of G, denoted by r(G) and diam (G) respectively. If these two are equal in a graph, that graph is called *self-centered* graph with radius r and is called an *r self-centered* graph. Such graphs are 2-connected graphs. A vertex u is said to be an eccentric vertex of v in a graph G, if d (u, v)=e(v) in that graph. In general, u is called an *eccentric vertex*, if it is an eccentric vertex of some vertex. For $v \in V(G)$, the *neighborhood* $N_G(v)$ of v is the set of all vertices adjacent to v in G. The set $N_G[v] = N_G(v) \cup \{v\}$ is called the *closed neighborhood* of v. A set S of edges in a graph is said to be independent if no two of the edges in S are adjacent. An edge e=(u, v) is a *dominating edge* in a graph G if every vertex of G is adjacent to at least one of u and v. For any set S of vertices in G, the induced sub graph <S> is the maximal sub graph with vertex set S. Also, a sub graph H of G is a component

Received: 3November, 2009; Revised: 14 January, 2010; Accepted: 5 March, 2010

of G if H is a maximal connected sub graph of G. The concept of distance and related properties are studied in [2], [3] and [14]. Also, the structural properties of some special class of graphs such as self centered graphs, radius critical graphs and eccentricity preserving spanning trees are studied in [4], [5], [8] and [10].

The concept of domination in graphs was introduced by Ore [13]. A set $D \subseteq V(G)$ is called dominating set of G if every vertex in V(G)-D is adjacent to some vertex in D and D is said to be a minimal dominating set if D-{v} is not a dominating set for any $v \in D$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set. We call a set of vertices a γ-set if it is a dominating set with cardinality $\gamma(G)$.Different types of dominating sets have been studied by imposing conditions on the dominating sets. A dominating set D is called *connected* (*independent*) *dominating set* if the induced sub graph <D> is connected (independent). D is called a *total dominating set* if every vertex in V(G) is adjacent to some vertex in D. A set D is called an *efficient dominating set* of G if every vertex in V-D is adjacent to exactly one vertex in D. A set $D \subseteq V(G)$ is called a *global dominating set* if D is a dominating set of G and $\overline{G}$. A set D is called a *restrained dominating set* if every vertex in V-D is adjacent to a vertex in D and another vertex in V-D. By γ_c, γ_i, γ_t, γ_e, γ_g and γ_r, we mean the minimum cardinality of a connected dominating set, independent dominating set, total dominating set, efficient dominating set, global dominating set and restrained dominating set respectively. The list of survey of domination theory papers are in [6], [7], [12], [15], [16] and [17].

The new concepts such as distance closed sets, distance preserving sub graphs, eccentricity preserving sub graphs, super eccentric graph of a graph, pseudo geodetic graphs are introduced and structural properties of those graphs are studied in [9]. Janakiraman and Alphonse [1] introduced and studied the concept of weak convex dominating sets, which mixes the concept of dominating set and distance preserving set. Using these, structural properties of various dominating parameters are studied. Continuing the above study, the concept of distance closed dominating set was defined and the structural properties of distance closed domination in various graphs are studied in [11].

In this paper, we introduce a new dominating set called even distance closed dominating set of a graph through which we studied the properties of the graph. We find upper and lower bounds of the new domination number in terms of various already known parameters. Also, we studied several interesting properties like Nordhaus-Gaddum type results relating the graph and its complement.

2. Prior Results

The concept of distance closed set is defined and studied in the doctoral thesis of Janakiraman [9] and the concept of distance closed sets in graph theory is due to the related concept of ideals in ring theory in algebra. The ideals in a ring are defined with respect to the multiplicative closure property with the elements of that ring. Similarly, the distance closed dominating set is defined with respect to the distance closed property and the dominating set of the graph. Thus, the distance closed dominating set of a graph G is defined as follows:

A subset $S \subseteq V(G)$ is said to be a distance closed dominating (D.C.D) set, if

(i) <S> is distance closed;

(ii) S is a dominating set.

The cardinality of a minimum D.C.D set of G is called the distance closed domination number of G and is denoted by γ_{dcl}.

Clearly from the definition, $1 \le \gamma_{dcl} \le p$ and graph with $\gamma_{dcl}=p$ is called a 0-distance closed dominating graph. Also, if S is a D.C.D set of G then the complement V-S need not be a D.C.D set of G. The definition and the extensive study of the above said distance closed dominating set in Graphs are studied in [11].

Following are some of the results related to the distance closed domination number of a graph presented in [11].

Theorem 2.1 [11]: If T is a tree with number of vertices $p \ge 2$, then $\gamma_{dcl}(T) = p-k+2$, where k is the number of pendant vertices in T.

Theorem 2.2 [11]: Let G be a Self centered graph of diameter 2.Then $\gamma_{dcl}(G) \le \delta+2$.

Theorem 2.3 [11]: Let G be a graph of order p. Then

(i) $\gamma_{dcl}(G)=2$ if and only if G has at least two vertices of degree p-1.

(ii) If G has exactly a vertex of degree p-1, then $\gamma_{dcl}(G)=3$.

Theorem 2.4 [11]: Let G be a graph of order p. If G has exactly a vertex of degree p-1, then $\gamma_{dcl}(G)=3$.

Theorem 2.5 [11]: If a graph G is connected and diam(G) ≥3, then $\gamma_{dcl}(\overline{G})=4$.

Theorem 2.6 [11]: For any connected graph G such that $\overline{G}$ is also connected, $\gamma_{dcl}(G)+\gamma_{dcl}(\overline{G}) \le p+4$, where $\gamma_{dcl}(G)$ and $\gamma_{dcl}(\overline{G})$are the cardinality of minimal distance closed dominating set of G and $\overline{G}$ respectively.

3. Main Results:

In this paper, we define a new domination parameter namely, even distance closed domination as follows.

Even distance closed dominating sets in Graphs:

A distance closed dominating set D is said to be an even distance closed dominating set (E.D.C.D), if for any vertex $v \in V\text{-}D$, there exists $v \in D$ at even distance from u.

The cardinality of minimum even distance closed dominating set is denoted by γ_{edcl}.

Theorem 3.1: There is no graph G, with $\gamma_{edcl}(G)=2$.

Proof: Suppose that, there is a graph G having an E.D.C.D set D with $|D|=2$. Then, clearly the two vertices in D are with eccentricity 1 and they adjacent to all the vertices of V-D. Hence, every vertex in V-D is at odd distance from D and hence D cannot be an E.D.C.D set of G.

Theorem 3.2: For any tree T, $\gamma_{edcl}= \gamma_{dcl}=p-k+2$.

Proof: Proof follows from Theorem 2.1[11].

Theorem 3.3: $\gamma_{edcl}(G)=3$ if and only if G satisfies the following two properties:

(i) G has exactly one vertex with eccentricity 1;

(ii) There exists two vertices u and v such that $N(u) \cap N(v)=\{x\}$, where x is a vertex with eccentricity 1.

Proof: From the first property (i), we have $\gamma_{dcl}(G)=3$.

Let x be a vertex with eccentricity 1. If there exists two vertices u and v such that $N(u) \cap N(v)=\{x\}$ then $D=\{u,x,v\}$ forms a D.C.D set of G and every vertex in V-D is at a distance 2 from u or v or both through x. Hence D is also an E.D.C.D set of G and hence $\gamma_{edcl}(G)=3$.

Conversely, assume that $\gamma_{edcl}(G)=3$. Suppose that, if there exists $y \in N(u) \cap N(v)$, $y \neq x$ then which only means that y is adjacent to both u and v. As u and v are arbitrary, $\gamma_{edcl}(G) \geq 4$, a contradiction to $\gamma_{edcl}(G)=3$.Thus, there exists two vertices u and v such that $N(u) \cap N(v)=\{x\}$.

Proposition 3.1: For any self centered graph of diameter 2, $\gamma_{edcl} \leq \delta+2$.

Proof: Let v be a vertex of G with degree δ. Then $D= v \cup N_1(v) \cup w$, where $w \in N_2(v)$ forms a D.C.D set of G and $\gamma_{dcl} \leq \delta+2$. Also any vertex $u \in V\text{-}D$ is a vertex of $N_2(v)$. This implies that $d(u,v)=2$. That is, D itself form an E.D.C.D set for G. Hence $\gamma_{edcl} \leq \delta+2$.

Remark 3.1: The above bound attained for C_5 and Petersen graph. Also, these graphs having the property that, every D.C.D set is an E.D.C.D set. Except these graphs, there are many 2-self centered graphs having this property (for example complete bipartite graph). Hence, we have the following theorem.

Theorem 3.4: If G is a 2-self centered graph and if for every $v \in V(G)$ both $\langle N_1(v) \rangle$ and $\langle N_2(v) \rangle$ are independent sets then, every D.C.D set of G is an E.D.C.D set of G.

Proof: Let G be a 2-self centered graph. Since, for every $v \in V(G)$ both $\langle N_1(v) \rangle$ and $\langle N_2(v) \rangle$ are independent, every vertex in $N_1(v)$ is adjacent to all the vertices of $N_2(v)$. Hence $\gamma_{dcl}=4$ and every D.C.D set D of G must contain at least one vertex from both $N_1(v)$ and $N_2(v)$, say $x \in N_1(v)$ and $y \in N_2(v)$. Then, every vertex in $N_2(v)$ is at a distance 2 from y and every vertex in $N_1(v)$ is at a distance 2 from x and also v is at a distance 2 from y. Hence D becomes an E.D.C.D set of G and hence every D.C.D set of G is an E.D.C.D set of G.

Theorem 3.5: If G is a 2-self centered graph, then a D.C.D set D of G is an E.D.C.D set of G, if for every vertex in $u \in V\text{-}D$, $E(u) \cap D \neq \Phi$.

Proof: Let G be a 2-self centered graph and let D be a D.C.D set of G. If D is also an E.D.C.D set of G then which only means that, every vertex in V-D is at a distance 2 from at least one vertex of D. That is, every vertex in V-D has at least one eccentric vertex in D as G is 2-self centered. Hence $E(u) \cap D \neq \Phi$.

Theorem 3.6: If G is a graph with radius 2 and diameter 3, then a D.C.D set D of G is an E.D.C.D set of G, if for every central vertex u in V-D, $E(u) \cap D \neq \Phi$.

Proof: Let G be a graph with radius 2 and diameter 3 and let D be a D.C.D set of G. If a vertex

$u \in V\text{-}D$, then we have the following cases:

Case 1: e(u)=3.

Every vertex with eccentricity 3 must be non adjacent to at least one vertex of D. For otherwise, if u is adjacent to all the vertices of D, then the eccentric node of u(say v) must be in V-D and v is non adjacent to all the vertices of D as d(u,v)=3, a contradiction to D is a D.C.D set of G. Hence u is non adjacent to at least one vertex of D and hence every vertex with eccentricity 3 must be at a even distance from at least one vertex of D.

Case 2: e(u)=2.

Suppose that, if a vertex with eccentricity 2 is adjacent to all the vertices of D then D can not be an E.D.C.D set of G. Thus, for D is also an E.D.C.D set of G if u is non adjacent to at least one vertex of D. That is, u has at least one eccentric node in D. Hence for every central vertex u in V-D, $E(u) \cap D \neq \Phi$.

Theorem 3.7: For any graph G with radius ≥2, $\gamma_{edcl} = \gamma_{dcl}$ if and only if for all minimum D.C.D set D in G, D-(D $\cap$N(u)) $\neq \Phi$, for all u$\in$V-D.

Proof: Suppose that $\gamma_{edcl} = \gamma_{dcl}$. To prove there exists a D.C.D set D in G such that for every vertex u$\in$V-D, D-(D $\cap$N(u)) $\neq \Phi$. If not, there exists no such D. Now, consider a minimal D.C.D set D of G. Then there exists a vertex u$\in$V-D such that D$\subseteq$N(u). This implies that u is adjacent to all the vertices of D. That is u is at odd distance from vertices of D. This implies that D itself can not form an E.D.C.D set for G. This is true for all minimal D.C.D set in G, which is a contradiction to $\gamma_{edcl} = \gamma_{dcl}$. Hence for every vertex u$\in$V-D, D-(D $\cap$N(u)) $\neq \Phi$.

Conversely, if there exists a minimal D.C.D set D such that D-(D $\cap$N(u)) $\neq \Phi$, for every vertex u$\in$V-D. Since D is a dominating set, every vertex u$\in$V-D is dominated by some vertex v and also γ_{dcl} ≥4, that is |D| ≥4 and u is not adjacent to all the vertices of D. Therefore, D itself forms an E.D.C.D set for G. Hence $\gamma_{edcl} = \gamma_{dcl}$.

Theorem 3.8: For any graph G, if $\gamma_{edcl} \neq \gamma_{dcl}$, then diameter of G is less than or equal to 4.

Proof: Let D be a minimal D.C.D set of G. If $\gamma_{edcl} \neq \gamma_{dcl}$ then by previous proposition, there exists a vertex u$\in$V-D such that all the vertices of D are adjacent to u. ---------------- (1)

Let x and y be any two non adjacent vertices of G

Case 1: If x and y are in D.

Then from (1), x and y are adjacent to u. This implies that d(x,y)=2.

Case 2: If x$\in$D and y$\in$V-D.

Since D is a dominating set, there exists a vertex v$\in$D such that y is adjacent to v and also from previous case d(x,v)≤ 2. This implies that d(x,y)≤ 3.

Case 3: If both x,y are in V-D.

Since D is a dominating set, x and y are dominated by some vertices x^1, y^1 in D respectively. From case 1, d(x^1,y^1)≤ 2. This implies that d(x,y)≤ 4. Hence the proof.

Theorem 3.9: If G is a graph with radius≥3, then every D.C.D set of G is an E.D.C.D set of G.

Proof: Let G be a graph with radius≥3 and let D be a D.C.D set of G.

Claim: D is also an E.D.C.D set of G.

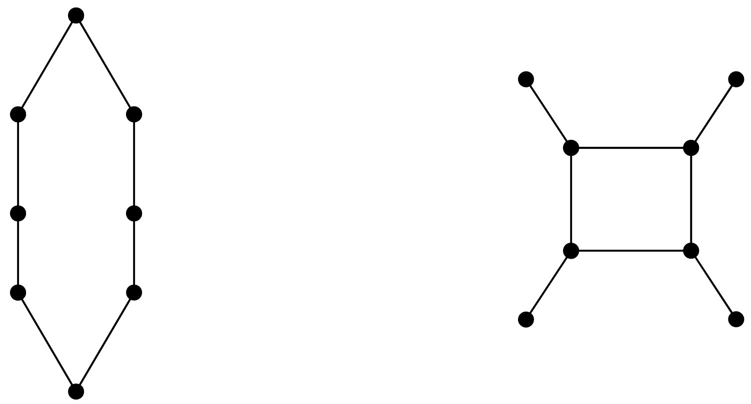
Suppose that, if a vertex u$\in$V-D is adjacent to all the vertices of D then e(u)=2(as every vertex of V-D is adjacent to at least one vertex of D), a contradiction to radius of G ≥3. Hence u is not adjacent to at least one vertex of D and hence D is also an E.D.C.D set of G.

Theorem 3.10: For any graph G with diameter greater than or equal to 3, $\gamma_{edcl}(G)+\gamma_{edcl}(\overline{G})\leq p+4$.

Proof: Let u and v be two vertices of G with distance greater than or equal to 3. Then clearly the edge uv in $\overline{G}$ will form a dominating set. Also the set {x,u,v,y} forms a D.C.D set of $\overline{G}$ where $x\in N_1(v)$ and $y\in N_1(u)$. In $\overline{G}$, every vertex of $N_1(u)$(in G) is at a distance 2 from u and every vertex of $N_1(v)$(in G) is at a distance 2 from v. Also, all the vertices which are not in $N_1[u]\cup N_1[v]$(in G) must be adjacent to both u and v in $\overline{G}$. Therefore, those vertices are at a distance 2 from both x and y in $\overline{G}$. Hence {x,u,v,y} itself forms an E.D.C.D set of $\overline{G}$ and hence $\gamma_{edcl}(\overline{G})= 4$.

Therefore, $\gamma_{edcl}(G)+\gamma_{edcl}(\overline{G})\leq p+4$.

Remark 3.2: The above bound is sharp and attainable. For example



For the above graphs, $\gamma_{edcl}(G)=8$ and $\gamma_{edcl}(\overline{G})=4$ and hence $\gamma_{edcl}(G)+\gamma_{edcl}(\overline{G})=12=p+4$.

That is, for any even cycles, paths and ciliates, the bound is sharp.

Theorem 3.11: Let G and $\overline{G}$ be self centered graphs of diameter 2 then, $\gamma_{edcl}(G)+\gamma_{edcl}(\overline{G})\leq p+3$.

Proof: We have $\gamma_{edcl}(G)\leq \delta+2$.

Thus, $\gamma_{edcl}(G)+\gamma_{edcl}(\overline{G}) \leq\delta(G)+2+\delta(\overline{G})+2$

$= \delta(G)+2+\Delta(G)+2$

$= \delta(G)+\Delta(G)+4$

$=p-1+4$

$=p+3$

Hence, $\gamma_{edcl}(G)+\gamma_{edcl}(\overline{G})\leq p+3$.

Remark 3.3: This bound is sharp and attainable for the graph C_5.

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