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Directed Edge–Graceful Labeling of the Graph $T_{t,n,m}$

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Abstract: *Rosa [14] introduced the notion of graceful labelings. The concept of magic, antimagic and conservative labelings have been extended to directed graphs [12]. Bloom and Hsu [3, 4, 5] extended the notion of graceful labeling to directed graph. In 1985, Lo [13] introduced the notion of edge - graceful graphs. We introduced [8] the concept of edge - graceful labelings to directed graphs and further studied in [9,10]. In this paper we investigate directed edge - graceful labeling of* $T_{t,n,m}$ *graph.*

Keywords: *Directed edge - graceful labeling, Directed edge - graceful graphs.*

AMS (MOS) Subject Classification: 05C78.

1. Introduction

All graphs in this paper are finite and directed. Terms not defined here are used in the sense of Harary [11]. The symbols $V(G)$ and $E(G)$ will denote the vertex set and edge set of a graph G. The cardinality of the vertex set is called the order of G denoted by p. The cardinality of the edge set is called the size of G denoted by q. A graph with p vertices and q edges is called a (p, q) graph.

A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. Labeled graphs serve as useful models for a broad range of applications such as coding theory, X–ray crystallography, radar, astronomy, circuit design, communication network addressing, database management etc. [1, 2]. A good account on graceful labeling problems and other types of graph labeling problems can be found in the dynamic survey of J.A. Gallian [6].

A graph G is called a graceful labeling if f is an injection from the vertices of G to the set $\{0, 1, 2, ..., q\}$ such that, when each edge xy is assigned the label $| f(x) – f(y) |$, the resulting edge labels are distinct.

A graph $G(V, E)$ is said to be edge - graceful if there exists a bijection f from E to $\{1, 2, ..., |E|\}$ such that the induced mapping f^+ from V to $\{0, 1, ..., |V| - 1\}$ given by, $f^+(x) = (\sum f(xy)) \bmod(|V|)$ taken overall edges xy incident at x is a bijection.

A necessary condition for a graph G with p vertices and q edges to be edge - graceful is $q(q + 1) \equiv \frac{p(p+1)}{2} (\bmod\ p)$. Bloom and Hsu [3, 4, 5] extended the notion of graceful labelling to directed graph. The concept of magic, antimagic and conservative

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labelings have been extended to directed graphs [12]. In [8] we extended the concept of edge - graceful labelings to directed graphs and further studied in [9, 10]. In this paper we investigate directed edge - graceful labeling of $T_{t,n,m}$ graph.

A (p, q) graph G is said to be **directed edge - graceful** if there exists an orientation of G and a labeling f of the arcs A of G with $\{1, 2, ..., q\}$ such that induced mapping g on V defined by, $g(v) = \left[f^+(v) - f^-(v) \right]$ (mod p) is a bijection where, $f^+(v)$ = the sum of the labels of all arcs with head v and $f^-(v)$ = the sum of the labels of all arcs with v as tail.

A graph G is said to be **directed edge - graceful graph** if it has directed edge–graceful labelings. Here, we investigate directed edge - graceful labeling of some trees and cycle related graphs.

2. Prior Results

Theorem 2.1: [8] The path P_{2n+1} is directed edge - graceful for all $n \geq 1$.

Theorem 2.2: [8] The cycle graph C_{2n+1} is directed edge - graceful for all $n \geq 1$.

Theorem 2.3: [8] The Butterfly graph B_n is directed edge - graceful if n is odd.

Theorem 2.4: [8] The Butterfly graph B_n is directed edge - graceful if n is even and $n \geq 4$.

Theorem 2.5: [8] The snail graph $SN(2n + 1)$ is directed edge - graceful for all $n \geq 1$.

Theorem 2.6: [8] $\left\langle K_{1,n} : K_{1,n} \right\rangle$ is directed edge - graceful if n is even and $n \geq 4$.

Theorem 2.7: [9] The graph $P_3 \cup K_{1,2n+1}$ is directed edge - graceful for all $n \geq 1$.

Theorem 2.8: [9] The graph P_{2m} @ $K_{1,2n+1}$ is directed edge-graceful for all $m \geq 2$ and $n \geq 1$.

Theorem 2.9: [9] The graph P_{2m+1} @ $K_{1,2n}$ is directed edge-graceful for all $m \geq 1$ and $n \geq 1$.

Theorem 2.10: [10] The fan F_{2n} is directed edge - graceful for all $n \geq 2$.

Theorem 2.11: [10] The star graph $K_{1,2n}$ is directed edge - graceful for all $n \geq 1$.

Theorem 2.12: [10] The wheel graph W_{2n} directed edge - graceful for all $n \geq 2$.

Theorem 2.13: [10] The tortoise graph T_{2n+1} is directed edge - graceful for all $n \geq 2$.

Theorem 2.14: [10] The graph nC_3 snake is directed edge - graceful for $n \geq 2$.

3. Main Results

Definition 3.1

$T_{t,n,m}$ is a graph obtained by joining the centers of $K_{1,n}$ and $K_{1,m}$ by a path P_t. It consists of $t + n + m$ vertices and $t + n + m - 1$.

Theorem 3.2

The graph $T_{t,n,m}$ is directed edge - graceful if t, n and m are odd numbers.

Proof

Let $G = T_{t,n,m}$ and $V[T_{t,n,m}] = \{w_1, w_2, ..., w_t, u_1, u_2, \ldots, u_n, v_1, v_2 \ldots, v_m\}$ be the set of vertices. Now we orient the edges of $T_{t,n,m}$ such that the arc set A is given by,

$A = \left\{(w_{2i-1}, w_{2i}),\ 1 \le i \le \frac{t-1}{2}\right\} \cup \left\{(w_{2i+1}, w_{2i}),\ 1 \le i \le \frac{t-1}{2}\right\} \cup \{(w_1, u_j),\ 1 \le j \le n\} \cup \{(w_t, v_k),$
$1 \le k \le m\}$

The edges and their orientation of $T_{t,n,m}$ are as in Fig. 1.

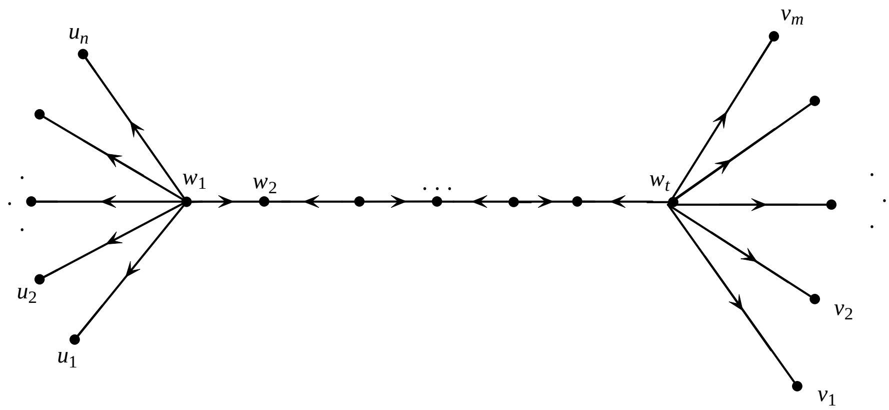

Fig. 1: $T_{t,n,m}$ with orientation

We now label the arcs of A as follows:

$f((w_{2i-1}, w_{2i})) = i,\ 1 \le i \le \frac{t-1}{2}$; $f((w_{2i+1}, w_{2i})) = \frac{t-1}{2} + n + m + i,\ 1 \le i \le \frac{t-1}{2}$

$f((w_1, u_1)) = \frac{t-1}{2} + n + m$; $f((w_t, v_1)) = \frac{t+1}{2}$

$f((w_1, u_{2j})) = \frac{t+m}{2} + j,\ 1 \le j \le \frac{n-1}{2}$; $f((w_1, u_{2j+1})) = \frac{t+m}{2} + n - j,\ 1 \le j \le \frac{n-1}{2}$

$f((w_t, v_{2k})) = \frac{t+1}{2} + k,\ 1 \le k \le \frac{m-1}{2}$; $f((w_t, v_{2k+1})) = \frac{t-1}{2} + n + m - k,\ 1 \le k \le \frac{m-1}{2}$

The values of $f^+(w_i)$, $f^+(u_j)$, $f^+(v_k)$ and $f^-(w_i)$, $f^-(u_j)$ and $f^-(v_k)$ are computed as under.

$f^+(w_1) = 0$; $f^-(w_1) = \frac{-1}{2}[n^2 + n(t+m+1) + m + 1]$

$f^+(w_{2i}) = \frac{t-1}{2} + n + m + 2i,\ 1 \le i \le \frac{t-1}{2}$; $f^-(w_{2i}) = 0$

$f^+(w_{2i+1}) = 0,\ 1 \le i \le \frac{t-3}{2}$; $f^-(w_{2i+1}) = -\left[\frac{t+1}{2} + n + m + 2i\right],\ 1 \le i \le \frac{t-3}{2}$

$f^+(w_t) = 0$; $f^-(w_t) = -\frac{1}{2}[m^2 + m(t+n+1) + 2t + n - 1]$

$f^+(u_1) = \frac{t-1}{2} + n + m$; $f^-(u_1) = 0$

$f^+(u_{2j}) = \frac{t+m}{2} + j,\ 1 \le j \le \frac{n-1}{2}$; $f^-(u_{2j}) = 0,\ 1 \le j \le \frac{n-1}{2}$

$f^+(u_{2j+1}) = \frac{t+m}{2} + \mathrm{n} - j,\ 1 \le j \le \frac{n-1}{2}$; $f^-(u_{2j+1}) = 0,\ 1 \le j \le \frac{n-1}{2}$

$f^+(v_1) = \frac{t+1}{2}$; $f^-(v_1) = 0$

$f^+(v_{2k}) = \frac{t+1}{2} + k,\ 1 \le k \le \frac{m-1}{2}$; $f^-(v_{2k}) = 0,\ 1 \le k \le \frac{m-1}{2}$

$f^+(v_{2k+1}) = \frac{t-1}{2} + n + m - k,\ 1 \le k \le \frac{m-1}{2}$; $f^-(v_{2k+1}) = 0,\ 1 \le k \le \frac{m-1}{2}$

Then the induced vertex labels are,

$g(u_1) = \frac{t-1}{2} + n + m$; $g(u_{2j}) = \frac{t+m}{2} + j,\ 1 \le j \le \frac{n-1}{2}$

$g(u_{2j+1}) = \frac{t+m}{2} + n - j,\ 1 \le j \le \frac{n-1}{2}$; $g(v_1) = \frac{t+1}{2}$

$g(v_{2k}) = \frac{t+1}{2} + k,\ 1 \le k \le \frac{m-1}{2}$; $g(v_{2k+1}) = \frac{t-1}{2} + n + m - k,\ 1 \le k \le \frac{m-1}{2}$

Case (i): $\frac{t-1}{2}$ **is odd**

$g(w_{2i-1}) = \frac{t+3}{2} - 2i,\ 1 \le i \le \frac{t+1}{4}$; $g(w_{2i}) = \frac{t-1}{2} + n + m + 2i,\ 1 \le i \le \frac{t-3}{4}$

$g\left(w_{\frac{t+1}{2}}\right) = 0$; $g\left(w_{\frac{t-1}{2}+2i}\right) = t + n + m + 1 - 2i,\ 1 \le i \le \frac{t+1}{4}$

$g\left(w_{\frac{t+1}{2}+2i}\right) = 2i,\ 1 \le i \le \frac{t-3}{4}$

Case (ii): $\frac{t-1}{2}$ **is even**

$g(w_{2i-1}) = \frac{t+3}{2} - 2i,\ 1 \le i \le \frac{t-1}{4}$; $g(w_{2i}) = \frac{t-1}{2} + n + m + 2i,\ 1 \le i \le \frac{t-1}{4}$

$g\left(w_{\frac{t+1}{2}}\right) = 0$; $g\left(w_{\frac{t-1}{2}+2i}\right) = 2i - 1,\ 1 \le i \le \frac{t-1}{4}$

$$g\left(w_{\frac{t+1}{2}+2i}\right) = t + n + m - 2i,\ 1 \leq i \leq \frac{t-1}{4}$$

Clearly, $g(V) = \{0, 1, 2, ..., (t + n + m - 1)\} = \{0, 1, 2, ..., p - 1\}$

So, it follows that all the vertex labels are distinct and g is a bijection. Hence, $T_{t,n,m}$ is a directed edge-graceful graph, if t, n and m are odd. The directed edge-graceful labeling of $T_{29,5,7}$ and $T_{31,5,7}$ is given in Fig. (2) and Fig. (3) respectively.

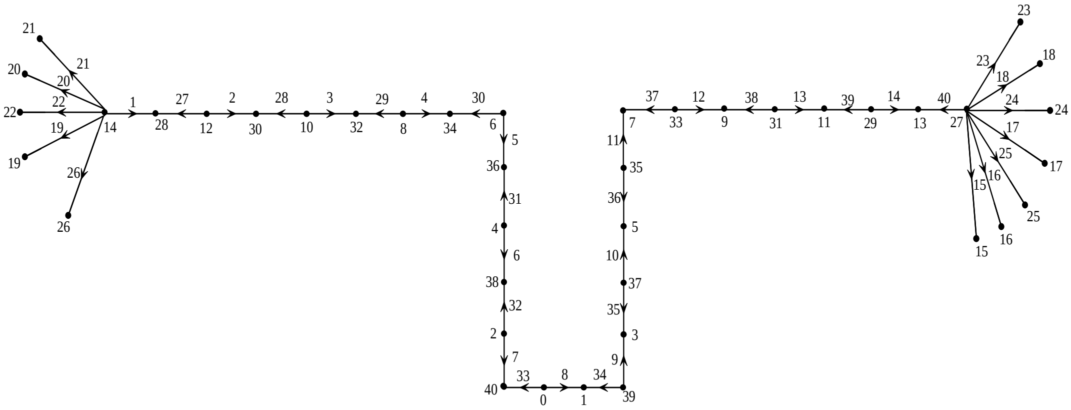

***Fig* (2) : $T_{29,5,7}$ with directed edge-graceful labeling**

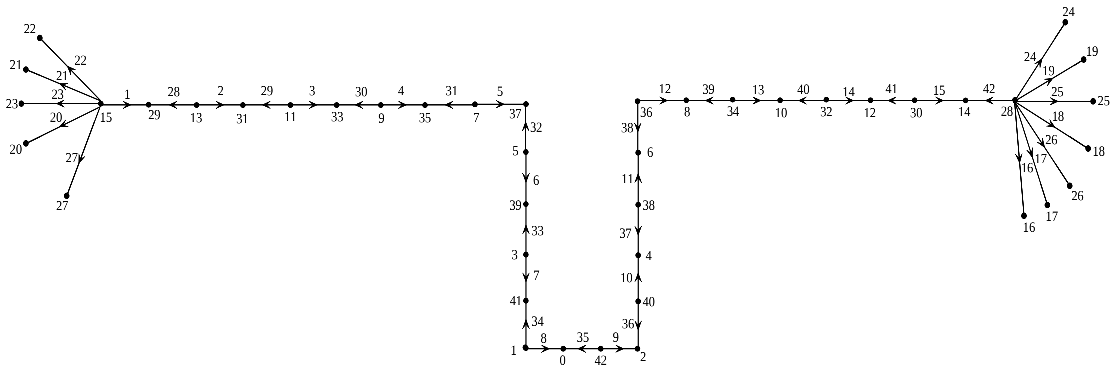

***Fig* (3) : $T_{31,5,7}$ with directed edge-graceful labeling**

Theorem 3.3

The graph $T_{t,n,m}$ is directed – edge graceful if t,n are even and m is odd.

Proof

Let $G = T_{t,n,m}$ and $V[T_{t,n,m}] = \{w_1, w_2, \ldots\ldots w_t, u_1, u_2, \ldots\ldots u_n, v_1, v_2, \ldots, v_m\}$ be the set of vertices. Now we orient the edges of $T_{t,n,m}$ such that the are set A is given by.

$$A = \left\{(w_{2i}, w_{2i+1}),\ 1 \leq i \leq \frac{t}{2}\right\} \cup \left\{(w_{2i}, w_{2i+1}),\ 1 \leq i \leq \frac{t}{2} - 1\right\} \cup \{(w_1, u_j), 1 \leq j \leq n\} \cup \{(w_t, v_k), 1 \leq k \leq m\}$$

The edges and their orientation of $T_{t,n,m}$ are as in *Fig* (4).

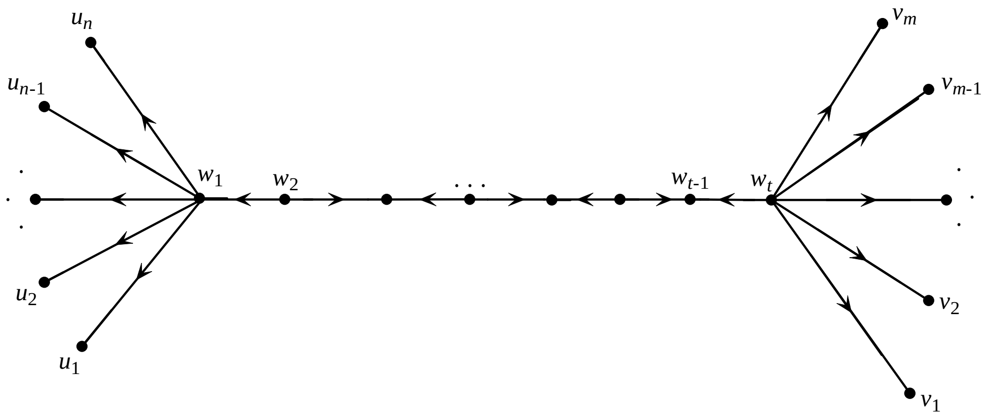

Fig. (4): $T_{t,n,m}$ with orientation

We now label the arcs of A as follows.

$f\left(\left(w_{2i},\ w_{2i+1}\right)\right) = i,\ 1 \le i \le \frac{t}{2} - 1;$ $\quad f\left(\left(w_{2i},\ w_{2i-1}\right)\right) = \frac{t}{2} + n + m - 1 + i,\ 1 \le i \le \frac{t}{2}$

$f\left(\left(w_1,\ u_{2j-1}\right)\right) = \frac{t}{2} - 1 + 2j,\ 1 \le j \le \frac{n}{2};$ $\quad f\left(\left(w_1,\ u_{2j}\right)\right) = \frac{t}{2} + n + m + 1 - 2j,\ 1 \le j \le \frac{n}{2}$

$f\left(\left(w_t,\ v_1\right)\right) = \frac{t}{2}$; $\quad f\left(\left(w_t,\ v_{2k}\right)\right) = \frac{t}{2} + \frac{n}{2} + 1 + 2k,\ 1 \le k \le \frac{m-1}{2}$

$f\left(\left(w_t,\ v_{2k+1}\right)\right) = \frac{1}{2}\left(t + n + m + 3\right) - 2k,\ 1 \le k \le \frac{m-1}{2}$

The values of $f^+(w_i)$, $f^+(u_j)$, $f^+(v_k)$ and $f^-(w_i)$, $f^-(u_j)$ and $f^-(v_k)$ are computed as under.

$f^+(w_1) = \frac{t}{2} + n + m$; $\quad f^-(w_1) = -\left[\frac{n}{2}\left(t + n + m\right)\right]$

$f^+(w_{2i}) = 0,\ 1 \le i \le \frac{t}{2} - 1$; $\quad f^-(w_{2i}) = -\left[\frac{t}{2} + n + m - 1 + 2i\right],\ 1 \le i \le \frac{t}{2} - 1$

$f^+(w_{2i+1}) = \frac{t}{2} + n + m + 2i,\ 1 \le i \le \frac{t}{2} - 1$; $\quad f^-(w_{2i+1}) = 0,\ 1 \le i \le \frac{t}{2} - 1$

$f^+(w_t) = 0$; $\quad f^-(w_t) = -\left[\frac{m^2}{2} + \frac{m}{2}\left(t + n + 1\right) + \left(t + \frac{n}{2} - 1\right)\right]$

$f^+(u_{2j-1}) = \frac{t}{2} - 1 + 2j,\ 1 \le j \le \frac{n}{2}$; $\quad f^-(u_{2j-1}) = 0,\ 1 \le j \le \frac{n}{2}$

$f^+(u_{2j}) = \frac{t}{2} + n + m + 1 - 2j,\ 1 \le j \le \frac{n}{2}$; $\quad f^-(u_{2j}) = 0,\ 1 \le j \le \frac{n}{2}$

$f^+(v_1) = \frac{t}{2}$; $\quad f^-(v_1) = 0$

$f^+(v_{2k}) = \frac{t}{2} + \frac{n}{2} + 1 + 2k,\ 1 \le k \le \frac{m-1}{2}$; $\quad f^-(v_{2k}) = 0,\ 1 \le k \le \frac{m-1}{2}$

$f^+(v_{2k+1}) = \frac{1}{2}\left(t + n + m - 1\right) + 2 - 2k,\ 1 \le k \le \frac{m-1}{2};$ $\quad f^-(v_{2k+1}) = 0,\ 1 \le k \le \frac{m-1}{2}$

Then the induced vertex labels are,

$g\,(u_{2j-1}) = \frac{t}{2} - 1 + 2j,\ 1 \le j \le \frac{n}{2}$; $g\,(u_{2j}) = \frac{t}{2} + n + m + 1 - 2j,\ 1 \le j \le \frac{n}{2}$

$g\,(v_1) = \frac{t}{2}$; $g\,(v_{2k}) = \frac{t}{2} + \frac{n}{2} + 1 + 2k,\ 1 \le k \le \frac{m-1}{2}$

$g\,(v_{2k+1}) = \frac{1}{2}(t + n + m - 1) + 2 - 2k,\ 1 \le k \le \frac{m-1}{2}$

Case (i): $\frac{t}{2}$ **is even**

$g\,(w_{2i-1}) = \frac{t}{2} + n + m - 2 + 2i,\ 1 \le i \le \frac{t}{4}$; $g\,(w_{2i}) = \frac{t}{2} + 1 - 2i,\ 1 \le i \le \frac{t}{4}$

$g\left(w_{\frac{t}{2}-1+2i}\right) = 2i - 2,\ 1 \le i \le \frac{t}{4}$; $g\left(w_{\frac{t}{2}+2i}\right) = t + n + m + 1 - 2i,\ 1 \le i \le \frac{t}{4}$

Case (ii) $\frac{t}{2}$ **is odd**

$g\,(w_{2i-1}) = \frac{t}{2} + n + m - 2 + 2i,\ 1 \le i \le \frac{t+2}{4}$; $g\,(w_{2i}) = \frac{t}{2} + 1 - 2i,\ 1 \le i \le \frac{t+2}{4}$

$g\left(w_{\frac{t}{2}+2i}\right) = 2i - 1,\ 1 \le i \le \frac{t-2}{4}$; $g\left(w_{\frac{t}{2}+1+2i}\right) = t + n + m - 2i,\ 1 \le i \le \frac{t-2}{4}$

Clearly, $g(V) = \{0, 1, \ldots, (t + n + m - 1)\} = \{0,1,\ldots, p - 1\}$

So, it follows that all the vertex labels are distinct and g is a bijection. Hence, $T_{t,n,m}$ is a directed edge - graceful graph if t, n are even and m is odd. The directed edge - graceful labeling of $T_{26,5,5}$ and $T_{28,6,5}$ is given in Fig. (5) and Fig. (6) respectively.

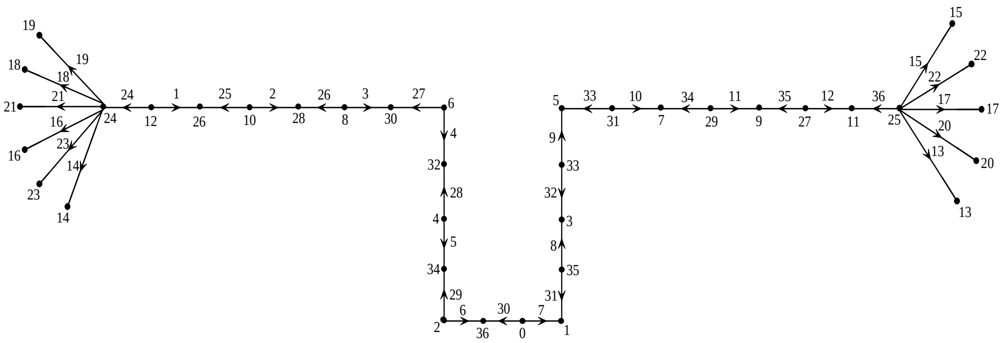

Fig (5) ; $T_{26,6,5}$ with directed edge-graceful labeling

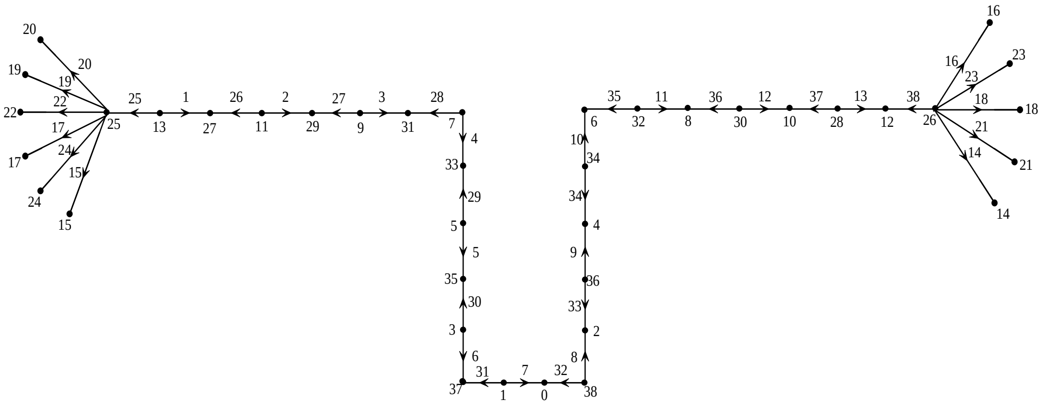

Fig (6) ; $T_{28,6,5}$ with directed edge-graceful labeling

Theorem 3.4

The graph $T_{t,n,m}$ is directed edge-graceful if n and m are even and t is odd.

Proof

Let $G = T_{t,n,m}$ and $V[T_{t,n,m}] = \{w_1, w_2, \ldots, w_t, u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_m\}$ be the set of vertices. Now we orient the edges of $T_{t,n,m}$ such that the are set A is given by,

$$A = \left\{(w_{2i-1}, w_{2i}),\ 1 \le i \le \frac{t-1}{2}\right\} \cup \left\{(w_{2i+1}, w_{2i}),\ 1 \le i \le \frac{t-1}{2}\right\} \cup \{(w_1, u_j), 1 \le j \le n\} \cup \{(w_t, v_k),$$

$1 \le k \le m\}$

The edges and their orientation of $T_{t,n,m}$ are as in Fig. (7).

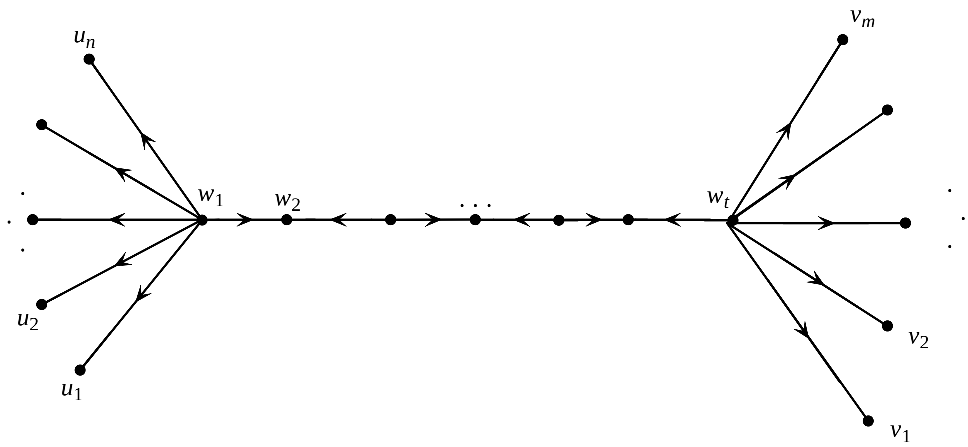

Fig. (7): $T_{t,n,m}$ with orientation

We now label the arcs of A as follows

$$f((w_{2i+1}, w_{2i})) = i,\ 1 \le i \le \frac{t-1}{2} \quad ; \quad f((w_{2i-1}, w_{2i})) = \frac{t-1}{2} + n + m + i,\ 1 \le i \le \frac{t-1}{2}$$

$$f((w_1, u_{2j-1})) = \frac{t-1}{2} - 1 + 2j,\ 1 \le j \le \frac{n}{2} \quad ; \quad f((w_1, u_{2j})) = \frac{t-1}{2} + n + m + 2 - 2j,\ 1 \le j \le \frac{n}{2}$$

$f((w_t, v_{2k-1})) = \frac{t-1}{2} + n - 1 + 2k,\ 1 \le k \le \frac{m}{2}$; $f((w_t, v_{2k})) = \frac{t-1}{2} + m + 2 - 2k,\ 1 \le k \le \frac{m}{2}$

The values of $f^+(w_i)$, $f^+(u_j)$, $f^+(v_k)$ and $f^-(w_i)$, $f^-(u_j)$, $f^-(v_k)$ are computed as under.

$f^+(w_1) = 0$; $f^-(w_1) = -\left[\frac{n^2}{2} + \frac{n}{2}(t+m+2) + \left(\frac{t+2m+1}{2}\right)\right]$

$f^+(w_{2i}) = \frac{t-1}{2} + n + m + 2i,\ 1 \le i \le \frac{t-1}{2}$; $f^-(w_{2i}) = 0,\ 1 \le i \le \frac{t-1}{2}$

$f^+(w_{2i+1}) = 0,\ 1 \le i \le \frac{t-3}{2}$; $f^-(w_{2i+1}) = -\left[\frac{t-1}{2} + n + m + 1\right] + 2i,\ 1 \le i \le \frac{t-3}{2}$

$f^+(w_t) = 0$; $f^-(w_t) = -\left[\frac{m}{2}(t+n+m) + \frac{t-1}{2}\right]$

$f^+(u_{2j-1}) = \frac{t-3}{2} + 2j,\ 1 \le j \le \frac{n}{2}$; $f^-(u_{2j-1}) = 0,\ 1 \le j \le \frac{n}{2}$

$f^+(u_{2j}) = \frac{t+3}{2} + n + m - 2j,\ 1 \le j \le \frac{n}{2}$; $f^-(u_{2j}) = 0,\ 1 \le j \le \frac{n}{2}$

$f^+(v_{2k-1}) = \frac{t-3}{2} + n + 2k,\ 1 \le k \le \frac{m}{2}$; $f^-(v_{2k-1}) = 0,\ 1 \le k \le \frac{m}{2}$

$f^+(v_{2k}) = \frac{t+3}{2} + m - 2k,\ 1 \le k \le \frac{m}{2}$; $f^-(v_{2k}) = 0,\ 1 \le k \le \frac{m}{2}$

Then the induced vertex labels are,

$g(u_{2j-1}) = \frac{t-3}{2} + 2j,\ 1 \le j \le \frac{n}{2}$; $g(u_{2j}) = \frac{t-3}{2} + n + m - 2j,\ 1 \le j \le \frac{n}{2}$

$g(v_{2k-1}) = \frac{t-3}{2} + n + 2k,\ 1 \le k \le \frac{m}{2}$; $g(v_{2k}) = \frac{t+3}{2} + m - 2k,\ 1 \le k \le \frac{m}{2}$

Case (i): $\frac{t-1}{2}$ is even

$g(w_{2i-1}) = \frac{t+3}{2} - 2i,\ 1 \le i \le \frac{t+3}{4}$; $g(w_{2i}) = \frac{t-1}{2} + n + m + 2i,\ 1 \le i \le \frac{t-1}{4}$

$g\left[w_{\frac{t-1}{2}+2i}\right] = 2i - 1,\ 1 \le i \le \frac{t-1}{4}$; $g\left[w_{\frac{t+1}{2}+2i}\right] = t + n + m - 2i,\ 1 \le i \le \frac{t-1}{4}$

Case (ii): $\frac{t-1}{2}$ **is odd**

$g(w_{2i-1}) = \frac{t+3}{2} - 2i,\ 1 \le i \le \frac{t+1}{4}$; $g(w_{2i}) = \frac{t-1}{2} + n + m + 2i,\ 1 \le i \le \frac{t-3}{4}$

$g\left[w_{\frac{t-3}{2}+2i}\right] = 2i - 2,\ 1 \le i \le \frac{t+1}{4}$; $g\left[w_{\frac{t-1}{2}+2i}\right] = t + n + m + 1 - 2i,\ 1 \le i \le \frac{t+1}{4}$

Clearly, $g(V) = \{0, 1, \ldots, (t + n + m - 1)\} = \{0, 1, \ldots, p - 1\}$

So, it follows that all the vertex labels are distinct and g is a bijection. Hence, $T_{t,n,m}$ is a directed edge - graceful graph if n and m are even and t is odd. The directed edge - graceful labeling of $T_{27,6,6}$ and $T_{29,6,6}$ is given in Fig. (8) and Fig. (9) respectively.

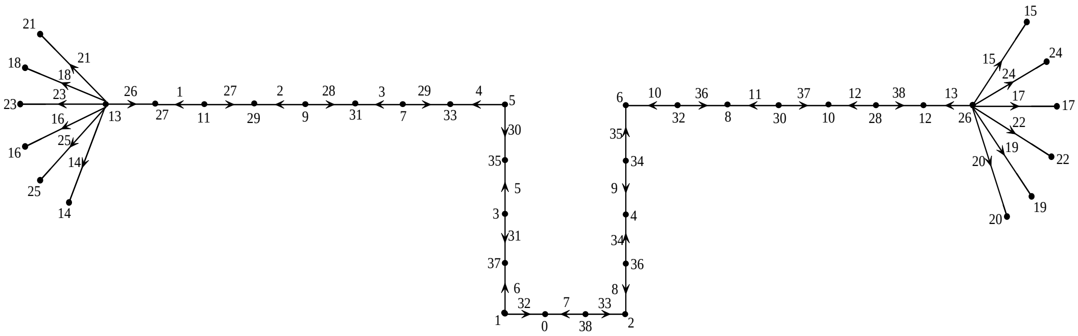

Fig (8) : $T_{27,6,6}$ with directed edge - graceful labeling

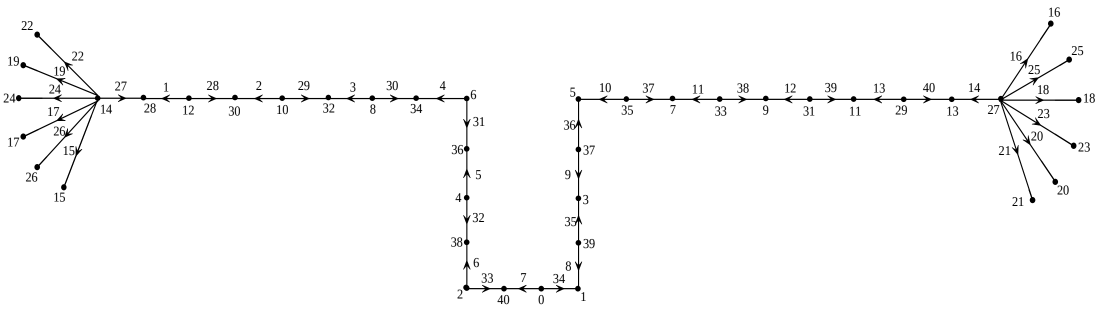

Fig (9) : $T_{29,6,6}$ with directed edge - graceful labeling

Theorem 3.5

The graph $T_{t,n,m}$ is directed edge - graceful if t, m are even and n is odd.

Proof

Let $G = T_{t,n,m}$ and $V[T_{t,n,m}] = \{w_1, w_2, \ldots, w_t, u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_m\}$ be the set of vertices. Now we orient the edges of $T_{t,n,m}$ such that the arc set A is given by,

$$A = \left\{(w_{2i-1}, w_{2i}), 1 \le i \le \frac{t}{2}\right\} \cup \left\{(w_{2i+1}, w_{2i}), 1 \le i \le \frac{t}{2} - 1\right\} \cup \{(w_1, u_j), 1 \le j \le n\} \cup \{(w_t, v_k), 1 \le k \le m\}$$

The edges and their orientation of $T_{t,n,m}$ are as in Fig. (10).

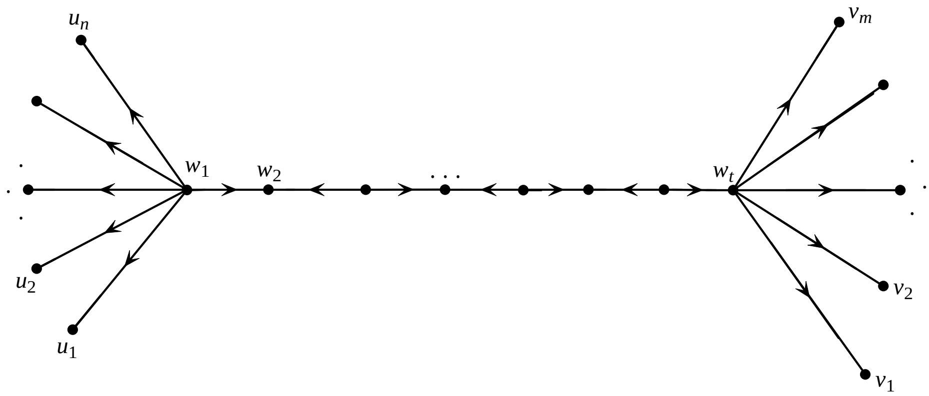

Fig. (10): $T_{t,n,m}$ with orientation

We now label the arcs of A as follows

$f\left(\left(w_{2i-1},\ w_{2i}\right)\right) = i,\ 1 \le i \le \frac{t}{2}$; $f\left(\left(w_{2i+1},\ w_{2i}\right)\right) = \frac{t}{2} + n + m + i,\ 1 \le i \le \frac{t}{2} - 1$

$f\left(\left(w_1,\ u_1\right)\right) = \frac{t}{2} + n + m$; $f\left(\left(w_1,\ u_{2j}\right)\right) = \frac{t}{2} - 1 + 2j,\ 1 \le j \le \frac{n-1}{2}$

$f\left(\left(w_1,\ u_{2j+1}\right)\right) = \frac{t}{2} + n + m + 1 - 2j,\ 1 \le j \le \frac{n-1}{2}$

$f\left(\left(w_t,\ v_{2k-1}\right)\right) = \frac{t}{2} + n - 2 + 2k,\ 1 \le k \le \frac{m}{2}$; $f\left(\left(w_t,\ v_{2k}\right)\right) = \frac{t}{2} + n + 3 - 2k,\ 1 \le k \le \frac{m}{2}$

The values of $f^+(w_i)$, $f^+(u_j)$, $f^+(v_k)$ and $f^-(w_i)$, $f^-(u_j)$ and $f^-(v_k)$ are computed as under.

$f^+(w_1) = 0$; $f^-(w_1) = -\frac{1}{2}\left[n^2 + n(t+m+1) + m + 2\right]$

$f^+(w_{2i}) = \frac{t}{2} + n + m + 2i,\ 1 \le i \le \frac{t}{2} - 1$; $f^-(w_{2i}) = 0,\ 1 \le i \le \frac{t}{2} - 1$

$f^+(w_{2i+1}) = 0,\ 1 \le i \le \frac{t}{2} - 1$; $f^-(w_{2i+1}) = -\left[\frac{t}{2} + n + m + 1 + 2i\right],\ 1 \le i \le \frac{t}{2} - 1$

$f^+(w_t) = \frac{t}{2}$; $f^-(w_t) = -\frac{m}{2}(t + n + m)$

$f^+(u_1) = \frac{t}{2} + n + m$; $f^-(u_1) = 0$

$f^+(u_{2j}) = \frac{t}{2} - 1 + 2j,\ 1 \le j \le \frac{n-1}{2}$; $f^-(u_{2j}) = 0,\ 1 \le j \le \frac{n-1}{2}$

$f^+(u_{2j+1}) = \left[\frac{t}{2} + n + m + 1 - 2j\right],\ 1 \le j \le \frac{n-1}{2}$; $f^-(u_{2j+1}) = 0,\ 1 \le j \le \frac{n-1}{2}$

$f^+(v_{2k-1}) = \left[\frac{t}{2} + n - 2 + 2k\right],\ 1 \le k \le \frac{m}{2}$; $f^-(v_{2k-1}) = 0,\ 1 \le k \le \frac{m}{2}$

$f^+(v_{2k}) = \frac{t}{2} + n + 3 - 2k,\ 1 \le k \le \frac{m}{2}$; $f^-(v_{2k}) = 0,\ 1 \le k \le \frac{m}{2}$

Then the induced vertex labels are,

$$g(u_1) = \frac{t}{2}+n+m \;;\quad g(u_{2j}) = \frac{t-2}{2}+2j,\ 1 \le j \le \frac{n-1}{2}$$

$$g(u_{2j+1}) = \frac{t}{2}+n+m+1-2j,\ 1 \le j \le \frac{n-1}{2} \;;\quad g(v_{2k-1}) = \frac{t}{2}+n-2+2k,\ 1 \le k \le \frac{m}{2}$$

$$g(v_{2k}) = \frac{t}{2}+n+3-2k,\ 1 \le k \le \frac{m}{2}$$

Case (i): $\frac{t}{2}$ is even

$$g(w_{2i-1}) = \frac{t}{2} + 1 - 2i,\ 1 \le i \le \frac{t}{4} \;;\quad g(w_{2i}) = \frac{t}{2} + n + m + 2i,\ 1 \le i \le \frac{t}{4} - 1$$

$$g\left(w_{\frac{t-4}{2}+2i}\right) = 2i - 2,\ 1 \le i \le \frac{t}{4} + 1 \;;\quad g\left(w_{\frac{t-2}{2}+2i}\right) = t + n + m + 1 - 2i,\ 1 \le i \le \frac{t}{4}$$

Case (ii): $\frac{t}{2}$ is odd

$$g(w_{2i-1}) = \frac{t}{2} + 1 - 2i,\ 1 \le i \le \frac{t+2}{4} \;;\quad g(w_{2i}) = \frac{t}{2} + n + m + 2i,\ 1 \le i \le \frac{t-2}{4}$$

$$g\left(w_{\frac{t-2}{2}+2i}\right) = 2i - 1,\ 1 \le i \le \frac{t+2}{4} \;;\quad g\left(w_{\frac{t}{2}+2i}\right) = t + n + m - 2i,\ 1 \le i \le \frac{t-2}{4}$$

Clearly, $g(V) = \{0, 1, \ldots, (t + n + m - 1)\} = \{0, 1, \ldots, p - 1\}$

So, it follows that all the vertex labels are distinct and g is a bijection. Hence, $T_{t,n,m}$ is a directed edge - graceful graph if t, m are even and n is odd. The directed edge - graceful labeling of $T_{28,5,6}$ and $T_{30,5,6}$ is given in Fig. (11) and Fig. (12) respectively.

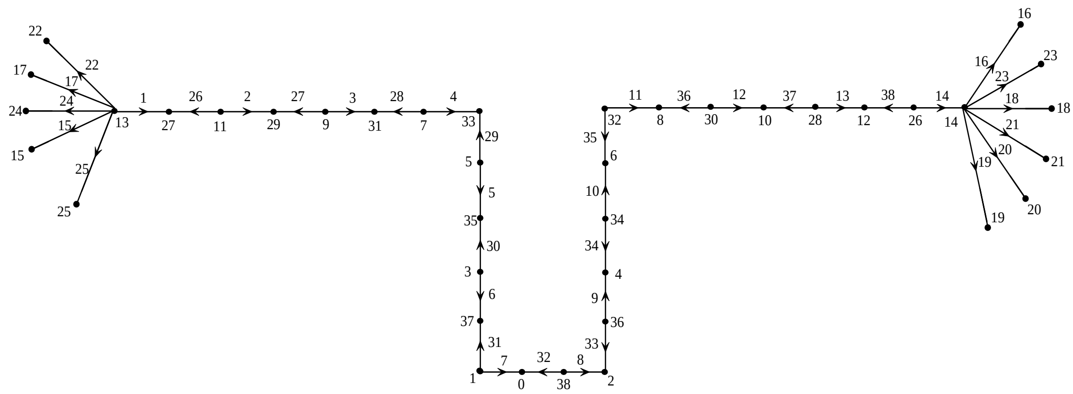

Fig (11) : $T_{28,5,6}$ with directed edge-graceful labeling

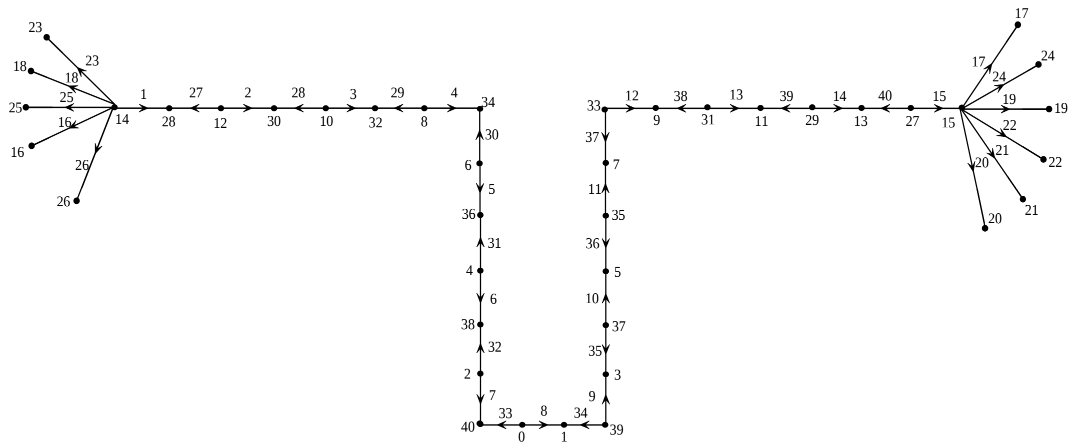

Fig (12) : $T_{30,5,6}$ with directed edge-graceful labeling

References

[1] G. S. Bloom and S. W. Golomb, Applications of numbered undirected graphs, *Proc–IEEE*, **65** (1977) 562 – 570.

[2] G. S. Bloom and S. W. Golomb, Numbered complete graphs, Unusual rulers, and assorted applications, in *Theory and Applications of Graphs*, *Lecture Notes in Math.*, **642**, Springer – Verlag, New York (1978) 53 – 65.

[3] G. S. Bloom and D. F. Hsu, On graceful digraphs and a problem in network addressing, *Congr. Numer.*, **35** (1982) 91 – 103.

[4] G. S. Bloom and D. F. Hsu, On graceful directed graphs that are computational models of some algebraic systems, *Graph Theory with Applications to Algorithms and Computers*, Ed. Y. Alavi, Wiley, New York (1985).

[5] G. S. Bloom and D. F. Hsu, On graceful directed graphs, *SIAM J. Alg. Discrete Meth.*, **6** (1985) 519 – 536.

[6] J. A. Gallian - A dynamic survey of graph labeling - *The electronic journal of combinatorics* (2010).

[7] B. Gayathri and M. Duraisamy, Even edge - graceful labeling of fan graphs, *Bulletin of Pure and Applied Sciences*, **26E** (No.2) (2007) 285 – 293.

[8] B. Gayathri and V. Vanitha, Directed edge - graceful labeling of some trees and cycle related graphs, *Bharathidasan University Journal of Science and Technology*, accepted for publication (in press)

[9] B. Gayathri and V. Vanitha, Directed edge - graceful labeling of path and star related trees, presented in the International Conference on Mathematics and Computer Science (ICMCS-2011), 7th and 8th January, 2011, Loyola College, Chennai.

[10] B. Gayathri and V. Vanitha Directed edge - graceful labeling of Some graphs, presented in the National Conference on recent advances in pure and applied mathematics (NCRAPAM-2011), 28th and 29th January 2011, Pudukkotai.

[11] F. Harary - Graph theory - Addison Wesley, Reading Mass (1972).

[12] S. Josephine Jeyarani, Studies in graph theory - labelings of graphs, Ph.D. thesis, Manonmaniam Sundaranar University, (1996).

[13] S. Lo - On edge - graceful labelings of graphs, *Congr. Numer.*, **50**(1985) 231 – 241.

[14] A. Rosa, On certain valuations of the vertices of a graph, *Theory of Graphs* (*Internat. Symposium*, *Rome*, *July 1966*), Gordon and Breac, N. Y. and Dunod Paris (1967) 349 – 355.