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On Super Duplicate Graphs

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Abstract: *For a graph G, let V(G) and E(G) denote its vertex set and edge set respectively. Let* $V'(G) = \{v' : v \in V(G)\}$ *be a copy of V(G). The Super duplicate graph D*(G) of G is the graph whose vertex set is* $V(G) \cup V'(G)$ *and edge set is* $E(\overline{G}) \cup \{u'v, uv' : uv \in V(G)\}$*, where* $\overline{G}$ *is the complement of G. In this paper, some basic properties of D*(G) are studied. Also a criterion for D*(G) to be Eulerian and a sufficient condition for Hamiltonicity are obtained. Finally, the parameters girth, connectivity, covering number, independence number, chromatic number, domination number and neighborhood number are determined for super duplicate graphs.*

1. Introduction

Graphs discussed in this paper are undirected and simple graphs. For a graph G, let V(G) and E(G) denote its vertex set and edge set respectively. *Eccentricity* of a vertex $u \in V(G)$ is defined as $e_G(u) = \max \{d_G(u, v): v \in V(G)\}$, where $d_G(u, v)$ is the distance between u and v in G. If there is no confusion, then we simply denote the eccentricity of vertex v in G as e(v) and d(u, v) to denote the distance between two vertices u, v in G respectively. The minimum and maximum eccentricities are the *radius* and *diameter* of G, denoted r(G) and diam(G) respectively. When diam(G) = r(G), G is called a *self-centered* graph with radius r, equivalently G is r-self-centered.

The concept of domination in graphs was introduced by Ore [4]. A set $D \subseteq V(G)$ is said to be a dominating set of G, if every vertex in $V(G) - D$ is adjacent to some vertex in D. D is said to be a minimal dominating set if $D - \{u\}$ is not a dominating set, for any $u \in D$. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set. D is a global dominating set, if it is a dominating set of both G and its complement $\overline{G}$. The global domination number γ_g of G is the minimum cardinality of a global dominating set [7]. A *total dominating set* D of G is a dominating set such that the induced sub graph <D> has no isolated vertices. The *total domination number* $\gamma_t(G)$ of G is the minimum

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cardinality of a total dominating set. This concept was introduced in Cockayne *et al* [1]. A γ-set is a minimum dominating set. Similarly, a γ_g-set, γ_t-set are defined.

For $v \in V(G)$, the neighborhood $N(v)$ of v is the set of all vertices adjacent to v in G. $N[v] = N(v) \cup \{v\}$ is called the closed neighborhood of v. A subset S of $V(G)$ is a *neighborhood set* (n-set) of G, if $G = \cup_{v \in S} \langle N[v] \rangle$, where $\langle N[v] \rangle$ is the sub graph of G induced by $N[v]$. The *neighborhood number* $n_0(G)$ of G is the minimum cardinality of an n-set of G [8].

For a graph G, let $V'(G) = \{v' : v \in V(G)\}$ be a copy of $V(G)$. Then the Duplicate graph $D(G)$ of G is the graph whose vertex set is $V(G) \cup V'(G)$ and edge set is $\{u'v$ and $uv' : uv \in E(G)\}$. This graph was first studied by Sampathkumar [6] and was further developed by Patil et al [5]. The *Super duplicate graph* $D^*(G)$ of G is the graph whose vertex set is $V(G) \cup V'(G)$ and edge set is $E(\overline{G}) \cup \{u'v, uv' : uv \in V(G)\}$, where $\overline{G}$ is the complement of G.

The concept of super duplicate graph of a given graph defines Boolean function of a graph based on the adjacency of the vertices of the given graph. The important application of facility location on networks is based on various types of graphical centrality, all of which are defined using distance. There has been rapid growth of research in the study of domination parameters of graphs, it is used in communication network, coding theory and in network surveillance by Radar stations; it finds application in Projective Geometry and in 'covering' or 'location problems'.

In this paper, some basic properties of $D^*(G)$ are studied. Also a criterion for $D^*(G)$ to be Eulerian and a sufficient condition for Hamiltonicity are obtained. Finally, the parameters girth, connectivity, covering number, independence number, chromatic number, domination number and neighborhood number are determined for super duplicate graphs. The definitions and details not furnished in this paper may found in [2].

2. Prior Results

In this section, we list some results with indicated references, which will be used in the subsequent main results.

Theorem 2.1[2]: For any nontrivial connected graph G, $\alpha_0 + \beta_0 = p = \alpha_1 + \beta_1$, where p is the number of vertices in G.

Theorem 2.2[5]: If G is either a complete graph $K_n(n \geq 3)$ or G contains an odd Hamiltonian cycle, then the Duplicate graph D(G) is Hamiltonian.

Proposition 2.3[8]: For a graph G of order p, neighborhood number n_0 of G is 1 if and only if G has a vertex of degree p-1.

Theorem 2.4 [3]: Let G be geodetic. Then $\overline{G}$ is also geodetic if and only if G is one of the following.

1. G is a cycle C_5 of length 5.
2. G is a path P_4 of length 3.
3. G is isomorphic to the Bull graph.

3. Main Results

The following elementary properties of a super duplicate graph are immediate. Let G be a (p, q) graph.

(a). Duplicate graph D(G) is spanning sub graph of D*(G). D*(G) is a (2p, ((p(p-1))/2) + q) graph.

(b). V(D*(G)) can be partitioned into two sets V and V' such that the sub graph of D*(G) induced by the vertices in V' is totally disconnected and that of D*(G) induced by the vertices in V is the complement of G.

(c). For any vertex $v \in V(G)$, there are two vertices v and v' in D*(G), such that $\deg_{D^*(G)}(v) = p - 1$ and $\deg_{D^*(G)}(v') = \deg_G(v)$. Hence, $\Delta(D^*(G)) = p - 1$ and $\delta(D^*(G)) = \delta(G)$.

(d). D*(G) is biregular with degree sequence k, p - 1 if and only if G is k- regular, where $k \neq p - 1$ and is regular if and only if G is complete.

(e). If G or $\overline{G}$ is non-planar, then D*(G) is also non-planar.

In the following, we find the graphs G for which D*(G) is disconnected.

Theorem 3.1: Let G be a graph with $\delta(G) \geq 1$. Then D*(G) is disconnected if and only if G is a complete bipartite graph.

Proof: Assume $\delta(G) \geq 1$ and D*(G) is disconnected. Since $\delta(G) \geq 1$ and $\overline{G}$ is an induced sub graph of D*(G), D*(G) is connected, if $\overline{G}$ is connected. Let $\overline{G}$ be disconnected and let $G_1, G_2, \ldots, G_n$, $(n \geq 2)$ be the components of $\overline{G}$. If one of them say,

G_1 is not complete, then there exists a vertex $v \in V(G_1)$ such that $v' \in V(D^*(G))$ is adjacent to at least one vertex in each component of $\overline{G}$ and hence in $D^*(G)$. Thus, $D^*(G)$ is connected. This is a contradiction. Hence, the components G_i $(1 \leq i \leq n)$ are complete in $\overline{G}$. Similarly, if $n \geq 3$, then for each $v \in V(G_i)$ $(1 \leq i \leq n)$, the vertex v' in $D^*(G)$ is adjacent to all the vertices in the remaining components. Hence, $D^*(G)$ is connected, which contradicts the fact that $D^*(G)$ is disconnected. Thus, $n = 2$ and G is a complete bipartite graph. Converse follows easily.

Remark 3.1: If $\delta(G) = 0$, then $\delta(D^*(G)) = 0$.

Lemma 3.1: $D^*(G)$ contains triangles if and only if either $\beta_0(G) \geq 3$ or G contains P_3; a path on 3 vertices as an induced sub graph, where $\beta_0(G)$ is the point independence number of G.

Proof: Assume $D^*(G)$ contains triangles. If $\overline{G}$ has triangles, then $\beta_0(G) \geq 3$. Let $\overline{G}$ be triangle-free. Since any two v_i' 's are nonadjacent in $D^*(G)$, any triangle in $D^*(G)$ will contain two vertices in $V(G)$ and one vertex in $V'(G)$. Then G contains P_3; a path on 3 vertices as an induced sub graph. Hence the lemma follows. Converse is obvious.

In the following, the solution for super duplicate graph, which is bipartite is obtained.

Theorem 3.2: For a connected graph G, $D^*(G)$ is bipartite if and only if G is complete.

Proof: Let $D^*(G)$ be bipartite. Then every cycle in $D^*(G)$ is of even length. By Lemma 3.1., if either $\beta o(G) \geq 3$ or G contains P_3 as an induced sub graph, then $D^*(G)$ is not bipartite. Hence, G is complete. Conversely, if G is complete, then $D^*(G)$ is bipartite with bipartition $[V(G), V'(G)]$.

Remark 3.2: By Theorem 3.2., it follows that $D^*(G)$ is regular bipartite if and only if G is complete.

A connected graph G is said to be ***geodetic***, if a unique shortest path joins any two of its vertices. In the following, the geodetic graphs G for which super duplicate graphs $D^*(G)$ are also geodetic are characterized.

Theorem 3.3: For any geodetic graph G with at least three vertices, $D^*(G)$ is geodetic if and only if G is either P_4; a path on four vertices or C_5; a cycle on five vertices.

Proof: Let G be any geodetic graph with at least three vertices. Then D*(G) is geodetic if $\overline{G}$ is geodetic, since $\overline{G}$ is an induced sub graph of D*(G). But $\overline{G}$ is geodetic if and only if G is one of the following graphs: P_4, C_5 and the Bull graph by Theorem 2.4. If G is the Bull graph, then D*(G) is not geodetic. Hence, G is either P_4 or C_5. Converse follows from the construction of D*(G).

Next, we prove that the girth of D*(G) is at most 6.

Theorem 3.4: For any connected graph G having at least three vertices, the girth of D*(G) is 3, 4 or 6.

Proof: By Lemma 3.1., if either $\beta_0(G) \geq 3$ or G contains P_3 as an induced sub graph, then D*(G) contains triangles and hence girth of D*(G) is 3. If not, then G is a complete graph. If G has least four vertices, then girth of D*(G) is 4. If $G \cong K_3$, then D*(G) is a cycle on six vertices and hence girth of D*(G) is 6.

Remark 3.3: Let G be a disconnected graph with $\beta_0(G) = 2$ and does not contain P_3 as an induced sub graph. Then girth of D*(G) is 4, since if G contains $C_3 \cup K_1$ as an induced sub graph or if

$G \cong 2K_2$, then D*(G) contains C_4 as an induced sub graph.

Theorem 3.5: D*(G) is not a tree, for any graph G.

Proof: Follows from Theorem 3.1., Theorem 3.4. and Remark 3.3.

In the following, a criterion for D*(G) being Eulerian is established.

Theorem 3.6: Let G be any (p, q) graph that is not complete bipartite and $\delta(G) \geq 1$. Then D*(G) is Eulerian if and only if p is odd and each vertex in G is of even degree.

Proof: Suppose D*(G) is Eulerian. Then the degree of the vertices v_i and v_i' of D*(G) are even. But $\deg(v_i)$ in D*(G) is p-1 and $\deg(v_i')$ in D*(G) is $\deg_G(v_i)$. Hence, p is odd and each vertex in G is of even degree. Conversely, assume that p is odd and each vertex in G is of even degree. Since G is not complete bipartite and $\delta(G) \geq 1$, D*(G) is connected. Further by the assumption, every vertex of D*(G) has even degree. Hence, D*(G) is Eulerian.

Theorem 3.7: D*(G) is Hamiltonian if G is either a complete graph or G contains an odd Hamiltonian cycle.

Proof: Since the duplicate graph D(G) of G is a spanning sub graph of $D^*(G)$, the theorem follows by Theorem 2.2.

Theorem 3.8: $D^*(K_n - e)$, ($n \geq 4$ is even) has a Hamiltonian path.

Proof: Let $v_1, v_2, .., v_n$ be the vertices of $K_n - e$, where $n \geq 4$ is even, such that v_2 be non-adjacent to v_4 (say). Then $v_2\, v_3'\, v_{4....}\, v_n\, v_1'\, v_{n-1}\, v_{n-2}' \ldots v_2'\, v_1\, v_n'$ is a Hamiltonian path in $D^*(K_n - e)$, where $n \geq 4$ is even.

Theorem 3.9: Let G be a graph with $\delta(G) \geq 2$ and $r(G) \geq 2$. Then each edge of $D^*(G)$ lies on a triangle iff

(i). For any two vertices u, v in G with $d_G(u, v) = 3$, $N_{\bar{G}}(u) \cap N_{\bar{G}}(v) \neq \phi$.

(ii). For each edge (u, v) in G, $N_{\bar{G}}(u) \cap N_G(v) \neq \Phi$ and $N_G(u) \cap N_{\bar{G}}(v) \neq \phi$.

Proof: Assume each edge of $D^*(G)$ lies on a triangle. $E(D^*(G)) = E(\bar{G}) \cup \{uv'$ and $u'v : uv \in E(G)\}$. Since $\bar{G}$ is an induced sub graph of $D^*(G)$, edges lying on a triangle in $\bar{G}$ also lie on a triangle in $D^*(G)$. Let e = (u, v) be an edge in $\bar{G}$ not lying in any triangle, where $u, v \in V(G)$. Then $d_G(u, v) \leq 3$. If $d_G(u, v) = 2$, then there exists a vertex $w \in V(G)$ adjacent to both u and v and $u\, w'\, v$ is a triangle in $D^*(G)$ and e lies on a triangle in $D^*(G)$. Since $(u, v) \notin E(G)$, $d_G(u, v) = 3$. Also since $V(D^*(G)) = V(G) \cup V'(G)$ and the duplicate graph D(G) is an induced sub graph of $D^*(G)$, no vertex in $V'(G)$ is adjacent to both u and v in $D^*(G)$. Thus, there must exist a vertex in $\bar{G}$ adjacent to both u and v and hence $N_{\bar{G}}(u) \cap N_{\bar{G}}(v) \neq \phi$. Consider the edge $e' = (u, v')$ in $D^*(G)$, where $(u, v) \in V(G)$. By the assumption, e' lies on a triangle in $D^*(G)$. Then there exists a vertex w in $D^*(G)$ adjacent to both u and v'. But $w \notin V'(G)$, since $\langle V'(G) \rangle$ is totally disconnected. Therefore, $w \in V(G)$ and hence $(u, w) \notin E(G)$ and $(v, w) \in E(G)$. Thus, $N_{\bar{G}}(u) \cap N_G(v) \neq \phi$. Similarly, the edge $e'' = (u', v)$ lies on a triangle in $D^*(G)$ implies that $N_G(u) \cap N_{\bar{G}}(v) \neq \phi$. Converse follows easily.

For a graph G, let $\kappa(G)$, $\lambda(G)$ and $\delta(G)$ denote respectively the vertex connectivity, edge connectivity and the minimum degree of G. We will use the theorem of Whitney [2]: For any graph G, $\kappa(G) \leq \lambda(G) \leq \delta(G)$.

Theorem 3.10: If G is a graph that is not complete bipartite, then $\kappa(D^*(G)) \leq \delta(G)$ and $\lambda(D^*(G)) \leq \delta(G)$.

Proof: Since $\delta(D^*(G)) = \delta(G)$, the proposition follows.

The bound $\kappa(D^*(G)) \leq \delta(G)$ is attained, when $r(G) = 1$ or G is a cycle on at least 4 vertices and the bound $\lambda(D^*(G)) \leq \delta(G)$ is attained, for all graphs G with $\delta(G) = 1$ and $G \cong K_4 - e$.

Remark 3.4: Let $\{v_i : 1 \leq i \leq t\}$, $(t \leq p)$ be a vertex cut of $\overline{G}$. If $< \overline{G} - \{v_i : 1 \leq i \leq t\} >$ contains exactly two complete components, then $\{v_i, v_i' : 1 \leq i \leq t\}$ is a vertex cut of $D^*(G)$.

In the following, point covering number α_0, point independence number β_0, line covering number α_1, line independence number β_1 and the chromatic number χ for $D^*(G)$ are determined.

Theorem 3.11: For any graph G with p vertices,

(i). $\alpha_0(D^*(G)) = p = \beta_0(D^*(G))$

(ii). $\alpha_1(D^*(G)) = 2\alpha_1(G)$ and $\beta_1(D^*(G)) = 2\beta_1(G)$.

Proof: Let V be the vertex set of G and V' be the set of new points introduced in the construction of $D^*(G)$. Since V' is an independent set with p points, $\beta_0(D^*(G)) \geq p$. Also each vertex in V is adjacent to at least one vertex in V' and hence any independent set in $D^*(G)$ can have at most p points. Thus, $\beta_0(D^*(G)) = p$. Since $D^*(G)$ has 2p points and $\alpha_0(D^*(G)) + \beta_0(D^*(G)) = 2p$, $\alpha_0(D^*(G)) = p$. It remains to prove that $\beta_1(D^*(G)) = 2\beta_1(G)$. It is to be observed that corresponding to each edge uv of G, there are two independent edges uv' and $u'v$ in $D^*(G)$. Thus, each edge of G gives rise to two independent edges in $D^*(G)$. So $\beta_1(G)$ independent edges of G give $2\beta_1(G)$ independent edges in $D^*(G)$ and this is the maximum number of independent edges in $D^*(G)$. Hence, $\beta_1(D^*(G)) = 2\beta_1(G)$. From the equation $\beta_1(D^*(G)) + \alpha_1(D^*(G)) = 2p = 2\beta_1(G) + 2\alpha_1(G)$, it follows that $\alpha_1(D^*(G)) = 2\alpha_1(G)$.

The ***clique cover number*** of G is the minimum number of complete sub graphs of G, needed to cover the vertices of G and is denoted by $\theta(G)$. For any simple graph G, $\chi(G) = \theta(\overline{G})$.

Theorem 3.12: For any graph G having no isolated vertices, $\chi(D^*(G)) = \theta(G)$ or $\theta(G) + 1$, where $\theta(G)$ is the clique cover number of G.

Proof: Since $\overline{G}$ is an induced sub graph of $D^*(G)$, $\chi(D^*(G)) \geq \chi(\overline{G})$ and hence $\chi(D^*(G)) \geq \theta(G)$, since $\chi(\overline{G}) = \theta(G)$.

Case(i): For each vertex $v \in V(G)$, $N_G(v)$ is k-colorable where $k < \chi(\overline{G})$.

Then color the vertex v' in $D^*(G)$ by a color other than k and thus, $D^*(G)$ is $\chi(\overline{G})$-colorable. Hence, $\chi(D^*(G)) \leq \chi(\overline{G}) = \theta(G)$. Therefore, $\chi(D^*(G)) = \theta(G)$.

Case(ii): There exists a vertex $v \in V(G)$ such that $N_G(v)$ is $\chi(\overline{G})(= \theta(G))$- colorable.

Since the sub graph of $D^*(G)$ induced by $\{v' : v \in V(G)\}$ is totally disconnected, $D^*(G)$ is $(\theta(G) + 1)$-colorable.

From Case (i) and Case(ii), it follows that $\chi(D^*(G)) = \theta(G)$ or $\theta(G) + 1$.

Example 3.1:

1. $\chi(D^*(K_n)) = 2$, if $n \geq 3$.
2. $\chi(D^*(C_n)) = 3$, if $n = 4$; and
   $= \chi(\overline{C_n})$, if $n \geq 5$
3. $\chi(D^*(K_{1,n})) = n + 1$, if $n \geq 2$.

In the following, a necessary and sufficient condition for a global dominating set of G to be a dominating set of $D^*(G)$ is found.

Theorem 3.13: Let G be any graph having no isolated vertices and D be a γ_g-set of G. Then $\gamma(D^*(G) \leq \gamma_g(G)$ if and only if $\delta(<D>) \geq 1$.

Proof: Let D be a γ_g-set of G and $\delta(<D>) \geq 1$. For any vertex v in G, there are two vertices v, v' in $D^*(G)$. Since $\overline{G}$ is an induced sub graph of $D^*(G)$, D dominates all the vertices in $V(\overline{G}) \cap V(D^*(G))$. By the assumption, for each vertex u' in $V'(G)$ there exists a vertex say, w in D such that u' is adjacent to w. Hence, D is a dominating set of $D^*(G)$. Thus, $\gamma(D^*(G)) \leq \gamma_g(G)$. Conversely, assume a γ_g-set D of G is also a dominating set of $D^*(G)$. If $\delta(<D>) = 0$, then there exists a vertex v in <D> such that $\deg_{<D>}(v) = 0$ and hence D does not dominate the vertex v'. This is a contradiction to the assumption. Hence, $\delta(<D>) \geq 1$.

This bound is attained, if $G \cong C_5$.

Remark 3.5: Theorem 3.13., can be restated as follows. Let D be a γ_g-set of a graph G having no isolated vertices. Then $\gamma(D^*(G)) \leq \gamma_g(G)$ if and only if D is a total dominating set of G.

Remark 3.6: In Theorem 3.13., D must be a γ_g-set of G since otherwise, if D is a dominating set of $\overline{G}$ but not of G, then there exists a vertex v in $\overline{G}$ adjacent to all the vertices in D and hence the vertex v' in $D^*(G)$ is not adjacent to any of the vertices in D.

Remark 3.7: Let D be a γ_g-set of G such that <D> contains isolated vertices. If $D' = \{v' \in V'(G): deg_D(v) = 0\}$, then $D \cup D'$ is a dominating set of $D^*(G)$.

Corollary 3.13.1: Let G be any graph having no isolated vertices and D be a γ_g-set of G. Then $\gamma_g(D^*(G)) \leq \gamma_g(G)$ if and only if $\delta(<D>) \geq 1$.

Corollary 3.13.2: $\gamma(D^*(G)) \leq \gamma_t(G)$ if and only if there exists a γ_t-set D of G such that for each $v \in V(G) - D$ there exists an $u \in D$ such that uv is not an edge in G, where $\gamma_t(G)$ is the total domination number of G.

Corollary 3.13.3: $2 \leq \gamma(D^*(G)) \leq p$.

The lower bound is attained, if $G \cong C_4$, $K_{1,n}$ or K_m, where $n \geq 2$ and $m \geq 3$ and the upper bound is attained, if $G \cong K_2$.

Theorem 3.14: $\gamma(D^*(G)) \leq \delta(G) + 1$, if the neighborhood set of a vertex of minimum degree is a dominating set of G.

Proof: Let $v \in V(G)$ be such that $deg_G(v) = \delta(G)$. If $N_G(v)$ is a dominating set of G, then $N_G[v]$ is a dominating set of $D^*(G)$ and hence $\gamma(D^*(G)) \leq \delta(G) + 1$.

This bound is attained, if $G \cong C_5$.

Theorem 3.15: If diam(G) = 2, then $\gamma(D^*(G)) \leq \delta(G) + 1$.

Proof: If diam(G) = 2, then $\gamma(G) \leq \delta(G)$. Let v be a vertex in G such that $deg_G(v) = \delta(G)$. Then N[v] is a dominating set of $D^*(G)$ and hence $\gamma(D^*(G)) \leq \delta(G) + 1$.

Corollary 3.15.1: If diam(G) = 2, then $\gamma_g(D^*(G)) \leq \delta(G) + 1$.

In the following, a condition for a neighborhood set (n-set) of $\overline{G}$ to be an neighborhood set of $D^*(G)$ is found, when G is triangle-free.

Theorem 3.16: Let G be any graph having no isolated vertices and be triangle-free. Then $n_0(D^*(G)) \leq n_0(\overline{G})$ if and only if there exists an n-set D of $\overline{G}$ with $|D| = n_0(\overline{G})$ such that D is a total dominating set of G.

Proof: Let D be an n-set of $\overline{G}$ with $|D| = n_0(\overline{G})$ such that D is a total dominating set of G. Then $D \subseteq V(D^*(G))$. Since D is an n-set of $\overline{G}$, it is enough to prove that the edges uv' and $u'v$ in $D^*(G)$ belong to $\cup_{w\in D}(E\langle N[w]\rangle)$.

(i). Since $\overline{G}$ is an induced sub graph of $D^*(G)$, the edges $x'y$, xy' in $D^*(G)$ with $x, y \in D$ and $xy \in E(G)$ belong to $\cup_{w\in D}(E\langle N[w]\rangle)$.

(ii). Let $x \in D$, $y \in V(G) - D$ such that $xy \in E(G)$. Since $x \in D$ the edge xy' in $D^*(G)$ belongs to $E(\langle N[x]\rangle)$. Since G is triangle-free and $\delta(\langle D\rangle) \geq 1$, there exists a vertex $z \in D$ such that $xz \in E(G)$ and $yz \notin E(G)$. Therefore, $x'z$, $yz \in E(D^*(G))$ and hence in $E(\langle N[z]\rangle)$. Thus, $x'y \in E(\langle N[z]\rangle)$.

(iii). Let $x, y \in V(G) - D$ such that $xy \in E(G)$. Since D is a dominating set of G, there exists a vertex $z \in D$, such that xz is an edge in G. Since G is triangle-free $yz \notin E(G)$. Therefore, the edges $x'z$, yz in $D^*(G)$ belongs to $E(\langle N[z]\rangle)$. Hence, $x'y \in E(\langle N[z]\rangle)$. Similarly, $xy' \in E(\langle N[z]\rangle)$ for some $z \in D$.

From (i), (ii) and (iii), it follows that each edge in $D^*(G)$ belongs to $\cup_{v\in D}(E\langle N[v]\rangle)$ and D is n-set for $D^*(G)$. Thus, $n_0(D^*(G)) \leq n_0(\overline{G})$. Conversely, assume any n-set D of $\overline{G}$ is also an n-set of $D^*(G)$. If D is not a dominating set of G, then for any two vertices x, $y \in V(G) - D$, with $xy \in E(G)$, the edges $x'y$ and xy' do not belong to $\cup_{z\in D}(E\langle N[z]\rangle)$. Similarly, if $\delta(\langle D\rangle) = 0$, then there exists a vertex z in D with $\deg_{\langle D\rangle}(z) = 0$. But since G has no isolated vertices and D is a dominating set of G, there exists a vertex $y \in V(G) - D$ adjacent to z. Hence, the edges $y'z$ and yz' are not in $\cup_{w\in D}(E\langle N[w]\rangle)$. Thus, $\delta(\langle D\rangle_G) \geq 1$.

Corollary 3.16.1: Let G be any graph having no isolated vertices and be triangle-free. If there exists an n-set D of $\overline{G}$ with $|D| = n_0(\overline{G})$ such that D is a point cover for G and $\delta(\langle D\rangle_G) \geq 1$, then D is also an n-set of $D^*(G)$.

In the following, a condition for an n-set of $\overline{G}$ to be an n-set of $D^*(G)$ is found, when G has triangles.

Theorem 3.17: Let G be any graph having no isolated vertices and contain triangles. Then $n_0(D^*(G)) \leq n_0(\overline{G})$ if and only if there exists an n-set D of $\overline{G}$ with $|D| = n_0(\overline{G})$ satisfying

(i). $\delta(\langle D\rangle_G) \geq 1$; and

(ii). For each edge uv in G, at least one of u and v is in $V(G)-D$, $N_G(u)\cap D \neq N_G(v)\cap D$, where both $N_G(u)\cap D$ and $N_G(v)\cap D$ contain at least one vertex in G.

Proof: Let D be an n-set of $\overline{G}$ with $|D| = n_0(\overline{G})$, satisfying conditions (i) and (ii). Since D is an n-set of $\overline{G}$ it is enough to prove the edges of the form xy', $x'y$ in $D^*(G)$ belong to $\cup_{w\in D}(E\langle N[w]\rangle)$, where $x, y\in V(G)$ and $xy\in E(G)$.

(a). For $x, y\in D$ with $xy\in E(G)$, the edges xy', $x'y$ in $D^*(G)$ belong to $\cup_{v\in D}(E\langle N[v]\rangle)$ since D is an n-set of $\overline{G}$.

(b). By conditions (i) and (ii), corresponding to the edge $xy\in E(G)$ with $x\in D$, $y\in V(G)-D$ or both $x, y\in V(G)-D$, the edges xy', $x'y$ in $D^*(G)$ belong to $\cup_{z\in D}(E\langle N[z]\rangle)$. Thus, each edge in $D^*(G)$ belongs to $\cup_{v\in D}(E\langle N[v]\rangle)$ and $D^*(G) = \cup_{v\in D}(E\langle N[v]\rangle)$. Hence, D is an n-set of $D^*(G)$ and $n_0(D^*(G)) \leq n_0(\overline{G})$. Conversely, assume any n-set D of $\overline{G}$ is also an n-set of $D^*(G)$. If condition (i) or (ii) is not true, then D cannot be an n-set of $D^*(G)$.

Corollary 3.17.1: Let G be any graph and D be a dominating set of $\overline{G}$ such that it is also a point cover for G. Then D is an n-set of $D^*(G)$ if and only if for each edge $uv\in E(G)$ with $u\in D$, $v\in V(G)-D$, $N_G(u)\cap D \neq N_G(v)\cap D$, where both $N_G(u)\cap D$ and $N_G(v)\cap D$ are not empty.

Theorem 3.18: For any graph G, $2 \leq n_0(D^*(G)) \leq p$.

Proof: $n_0(D^*(G)) = 1$ if and only if $D^*(G)$ has a vertex of degree (2p-1). Hence, $n_0(D^*(G)) \geq 2$. Also the set V(G) is an n-set for $D^*(G)$ and hence $n_0(D^*(G)) \leq p$.

The lower bound is attained, if G is a star and the upper bound is attained, if G is a complete graph on at least 2 vertices.

Example 3.2:

1. $n_0(D^*(C_n)) = 3$, if $n = 3$; and
   $= n - 2$, if $n \geq 4$.
2. $n_0(D^*(K_{m,n})) = 2$, if $m, n \geq 2$.
3. If G is the wheel on n vertices, then $n_0(D^*(G)) = n - 2$, if $n \geq 5$.

References

[1] E. J.Cockayne, R.M. Dawes and S.T. Hedetniemi, Total domination in graphs, Networks, 10 (1980), 211-219.

[2] F. Harary, Graph Theory, Addison- Wesley, Reading Mass, (1972).

[3] T. N. Janakiraman, (1991), Eccentricity properties of some classes of graphs, Ph.D. Thesis, Madras University, Tamil Nadu.

[4] O. Ore, Theory of graphs, Amer. Math. Soc. Colloq. Publ., 38, Providence, (1962).

[5] H. P. Patil and S. Thangamari, Miscellaneous properties of Splitting graphs and related concepts, Proceedings of the national workshop on Graph Theory and its applications, Manonmaniam Sundaranar Uni, Tirunelveli, Feb 21-27, 996, pp 121-128.

[6] E. Sampathkumar, On Duplicate Graphs, J. Indian Math. Soc., 37 (1973), 285-293.

[7] E. Sampathkumar, The global domination number of a graph, J. Math. Phy. Sci. **23**(5), 1989.

[8] E. Sampathkumar and Prabha S. Neeralagi, The neighborhood number of a graph, Indian J. pure appl. Math., 16(2): 126-132, February 1985.