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Just Excellent Graphs

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Abstract: *A graph G is said to be excellent, if every vertex of G belongs to a γ - set. In this paper, we introduce a new class of graphs, called just excellent graphs and initiate a study on this class. A graph G is said to be just excellent if to each u $\in$ V, there is a unique γ - set of G containing u. We obtain a necessary and sufficient condition for a graph to be just excellent. We find an upper bound for the domination number of a just excellent graph. If G is just excellent and γ (G) attains this upper bound, then we show that G is Hamiltonian. We show that every just excellent graph contains no cut vertex. We also prove that every graph is an induced subgraph of a just excellent graph.*

Key words: Domination, Level vertex, Excellent graph, Just excellent graph.

Introduction

We consider only simple undirected graphs. For graph theoretic terminologies we refer to [1]. Let $G = (V, E)$ be a graph. A set $D \subseteq V$ is a dominating set if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set. A dominating set with cardinality $\gamma(G)$ is called γ - set of G. For information on domination refer to [2] and [3].

A graph G is said to be excellent, if every vertex of G belongs to a γ - set. The domatic number $d(G)$ of a graph G is defined to be the maximum number of elements in a partition of $V(G)$ into dominating sets.

Let $u \in V(G)$. Then $\gamma^u(G, u) = min \{ |S| : S \subseteq V, S \text{ dominates } G - u \}$. By a $\gamma^u(G, u)$ - set we mean a set $S \subseteq V$, with $|S| = \gamma^u(S, G)$, which dominates $G - u$. For a vertex $u \in V(G)$

1. If $\gamma^u(G, u) = \gamma(G)$, then u is said to be a γ - level vertex of G, or simply a level vertex of G.
2. If $\gamma^u(G, u) = \gamma(G) - 1$, then u is said to be a γ - no level vertex of G, or simply a nonlevel vertex of G.

The private neighbor set of a vertex v in a γ - set S, denoted by $PN(v, S)$ is $N(v) - N[S - \{v\}]$ and each $u \in PN(v, S)$ is called the private neighbor of v with respect to S.

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Definition:

A graph G is said to be just excellent if to each $u \in V$, there is a unique γ - set of G containing u.

Remarks:

1. *Every just excellent graph is excellent.*
2. *If G is just excellent and $G \neq K_n$, there is no vertex u, such that $N[u]$ is a clique.*

   Proof:

   If there exist a vertex u such that $N[u]$ is complete, then consider the set γ – set S of G that contains u. Then $(S - u) \cup \{v\}$ is also a γ - set of G, for every $v \in N(u)$. Hence there are atleast two γ - sets of G containing elements of $S - u$, which is a contradiction.
3. *If G is just excellent, then* $\delta(u) \geq \frac{n}{\gamma(G)} - 1$.

   Proof:

   Let $V = S_1 \cup S_2 \cup ... \cup S_m$ be the partition of V into γ - sets of G. Fix one $u \in V$. Assume that $u \in S_j$. Since each S_i is a γ - set, u is adjacent to atleast one vertex of S_i, $i \neq j$. Hence $\delta(u) \geq m - 1 = \frac{n}{\gamma(G)} - 1$.
4. *If $G \neq K_2$, $\overline{K_n}$ is just excellent, then $\delta(G) \geq 2$. [In particular any tree $T \neq K_2$ is not just excellent.]*

   Proof:

   Assume that $G \neq K_2$, $\overline{K_n}$, and u be a pendant vertex of G. Let $N(u) = \{v\}$. Since G is just excellent, there exists a γ - set D of G containing u. As $u \in D$, $v \notin D$. As $G \neq K_2$ and $u \in D$, $|D| \geq 2$. So $(D-u) \cup \{v\}$ is a γ - set of G, which is a contradiction as G is just excellent.
5. *Every just excellent graph $G \neq \overline{K_n}$ is connected.*

   Proof:

   If G is not connected, (and as $G \neq \overline{K_n}$), one of the connected components G_1, of G contains more than one vertex. As G_1 is also just excellent, and $\gamma(G_1) \leq \frac{|G_1|}{2}$, G_1 has more than one γ - set. Select two γ - sets S_1 and S_2 of G_1. Fix one γ - set D for $G - G_1$. Then both $D \cup S_1$, and $D \cup S_2$ are γ - sets of G, which is a contradiction as G is just excellent. Hence every just excellent graph is connected.

6. *If $G \neq \overline{K_n}$ is just excellent, then $|PN(u, D)| \geq 2$ for all $u \in D$, where D is any γ- set of G.*

   Proof:

   Let D be a γ - set of G. If $PN(u, D) = \phi$, $(D - u) \cup \{w\}$ is also a γ - set, for any $w \in N(u)$. If $|PN(u, D)| = 1$, let $PN(u, D) = \{w\}$. Then $(D - u) \cup \{w\}$ is a γ - set of G. In either case, we get a contradiction as G is just excellent. So $|PN(u,D)| \geq 2$, $\forall\ u \in D$.

7. *If G, $G \neq \overline{K_n}$ is just excellent, then $\Delta(G) \leq n - 2k + 1$, where $k = \gamma(G)$.*

   Proof:

   Let $u \in V(G)$. Let S be a γ - set for G which contains u. $|PN(w, S)| \geq 2$, $\forall\ w \in S$. So u is not adjacent to any of the vertices in $\bigcup_{w \neq u \in S} PN(w, S)$ and $deg(u) \leq (n - 1) - 2(|S| - 1)$. As this is true for all $u \in V(G)$, $\Delta \leq n-2k+1$.

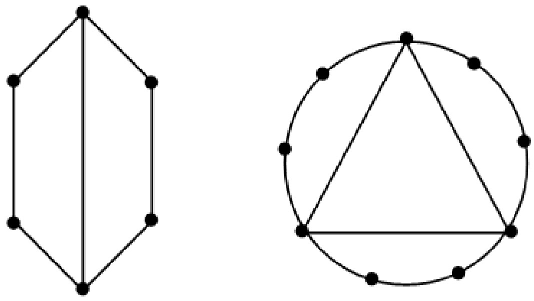


Fig. 1 Examples of graphs for which $\Delta = n - 2k + 1$.

Examples of graphs which are just excellent

1. Every cycle C_{3n} is just excellent.
2. Every complete graph K_n is just excellent.
3. Given a graph G with $\delta \geq 1$, a graph denoted by G_I is obtained as follows. To each $u \in V(G)$, a clique A_u of order $deg_G(u)$ is obtained, and a bijection $\phi_u : N(u) \rightarrow A_u$ is constructed. $\phi_u(v)$ is denoted by v', $\forall\ v \in N(u)$.

   Now $V(G_I) = \bigcup E(A_u) \cup \{u'v' | uv \in E(G), v' \in A_u, u' \in A_v\}$

   Then $|V(G_I)| = 2|E(G)|$ and $|E(G_I)| = \frac{1}{2} \sum_{u \in V} (\deg(u))^2$. The graph G_I is known as the inflated graph of G. G_I is just excellent when $G = K_n$ or C_{3n}. Also $\gamma(G_I) = n - 1$ *if* $G = K_n$, $(n \geq 2)$ and $\gamma(G_I) = 2n$ if $G = C_{3n}$.

4. I_n. Let $O_n = u_0, u_1, \ldots, u_{n-1}, u_0$ and $I_n = u_0', u_1', \ldots, u_{n-1}', u_0'$. We combine the cycles O_n and I_n and obtain a graph denoted by Y_n, where $E(Y_n) = \{u_i u_i', u_i u_{i+1}' \mid 0 \le i \le n-1\} \cup E(O_n) \cup E(I_n)$. The graphs Y_5, Y_{10}, Y_{15}, ... are just excellent.

In the following theorem, we obtain a necessary and sufficient condition for a graph to be just excellent.

Theorem 1:

The graph G is just excellent if and only if

1. *$\gamma(G)$ divides n.*
2. $d(G) = \dfrac{n}{\gamma(G)}$
3. *G has exactly $\dfrac{n}{\gamma(G)}$ distinct γ- sets.*

Proof:

Let G be just excellent. Let $S_1, S_2, \ldots, S_m$ be the collection of distinct γ- sets of G. Since G is just excellent, these sets are pair wise disjoint and their union is $V(G)$. So $V = S_1 \cup S_2 \cup \ldots \cup S_m$ is a partition of V into γ- sets of G. Since $|S_i| = \gamma(G)$, $\forall\ i = 1, 2, \ldots, m$ we have

1. domatic number of G = m, and
2. $m\,\gamma(G) = n$.

So both $\gamma(G)$ and $d(G)$ are divisors of n and $d(G) = \dfrac{n}{\gamma(G)}$. Also, G has exactly $m = \dfrac{n}{\gamma(G)}$ distinct γ- sets. Conversely, assume G to be a graph satisfying the hypothesis of the theorem. Let $m = \dfrac{n}{\gamma(G)}$. Let $V = S_1 \cup S_2 \cup \ldots \cup S_m$ be a decomposition of dominating sets of G. Now as $\gamma(G)m = n = \sum_{i=1}^{m} |S_i| \ge m\gamma(G)$, for each i, S_i is a γ- set of G. Since it is given that G has exactly m distinct γ- sets, $S_1, S_2, \ldots, S_m$ are the distinct γ- sets of G. Since $V = S_1 \cup S_2 \cup \ldots \cup S_m$ is a partition, each vertex of V belongs to exactly one S_i. Hence G is just excellent.

Theorem 2:

Let G $\neq K_2$ be just excellent. Then $\gamma(G) \le \dfrac{n}{3}$.

Proof:

Let D be a γ - set of G. Then by remark 6, $|\ PN(\ u,\ D\ )\ | \geq 2$, $\forall\ u \in D$. If $G \neq K_n$ is just excellent, then $d(\ G\ ) \geq 2$. If possible assume that $d\ (\ G\ ) = 2$. Then $V = S_1 \cup S_2$, where S_1 and S_2 are the distinct γ - sets of G. As $|\ PN\ (u,\ S_1\ )\ | \geq 2$, $\forall\ u \in S_1$ and as $PN(\ u,\ S_1\ ) \subseteq S_2$ we get that $|\ S_2\ | \geq 2|\ S_1\ | = 2\gamma(\ G\ )$. But $|\ S_1\ | = |\ S_2\ |$. Hence $d(G) \geq 3$. Since G is just excellent, $n = \gamma(\ G\ )\ d(\ G\ )$. As $d(\ G\ ) \geq 3$, we get that $\gamma(G) \leq \frac{n}{3}$.

Remark:

The bound in the above theorem is sharp since $\gamma(\ C_{3n}\ ) = n$, $\forall\ n \geq 1$.

We now prove that the just excellent graphs for which the upper bound for $\gamma(\ G\ )$ is attained are Hamiltonian.

Theorem 3:

If G is just excellent and $\gamma(G) = \frac{n}{3}$, *then G is Hamiltonian.*

Proof:

If $\gamma(G) = \frac{n}{3}$ and G is just excellent, the domatic number $d\ (\ G\ ) = 3$. Let S_1, S_2, S_3 be the distinct γ - sets of G. Then $V\ (\ G\ ) = S_1 \cup S_2 \cup S_3$. Let $A = \{e \in E\ (\ G\ )\ |\ e \in \langle S_i \rangle\ \}$ for some $i = 1, 2, 3\}$. Then if $H = G - A$, then $\gamma(\ G\ ) \leq \gamma(\ H\ )$. As S_1, S_2, S_3 are the dominating sets of H, $\gamma(\ G\ ) = \gamma(\ H\ )$. The sets S_1, S_2, S_3 are the distinct γ - sets of H, and H is just excellent. If H is Hamiltonian, then G is so. Hence it is enough to prove the result by assuming that each S_i is an independent set in G.

By remark 6, if $u \in S_i$, then $|\ PN(\ u,\ S_i\ )\ | \geq 2$ and as $u \neq v \in S_i \Rightarrow PN(\ u,\ S_i\ )$ and $PN(\ v,\ S_i\ )$ are disjoint, it follows that $\bigcup_{u \in S_i} PN(u, S_i) \geq 2\gamma(G) = |V - S_i|$. [Since $|\ V - S_i\ | = n - \gamma(G) = 3\gamma(G) - \gamma(G) = 2\gamma(G)$]. Hence $deg\ (u) = 2$, $\forall u \in V\ (G)$. As $\gamma(G) = \frac{n}{3}$, $G \neq \overline{K_n}$, and as G is just excellent, G is connected. Since G is connected and 2 - regular, it is a cycle C_n.

Theorem 4:

Every just excellent graph G, contains no cut vertex and hence if G $\neq K_2$, *it contains no bridge.*

Proof:

If possible assume that G has a cut vertex u. Let H_1 be a component of $G - u$. Let G_1 be the induced subgraph $<\ H_1 \cup \{u\}\ >$ of G and let $G_2 = G - H_1$. Then clearly

$\gamma(G) \leq \gamma(G_1) + \gamma(G_2)$. Let S be the γ- set of G that contains u. Let D be a γ- set of G not containing u. [*As* $G \neq \overline{K_n}$ and G is just excellent, there is a γ- set of G not containing u]. Let $S_i = S \cap V(G_i)$ and $D_i = D \cap V(G_i)$, $i = 1, 2$. Since S_i dominates G_i, $|S_i| \geq \gamma(G_i)$. Now, $\gamma(G) = |S| = |S_1| + |S_2| - 1 \geq \gamma(G_1) + \gamma(G_2) - 1$.

Thus, $\gamma(G_1) + \gamma(G_2) - 1 \leq \gamma(G) \leq \gamma(G_1) + \gamma(G_2)$ ... (1).

Assume that $\gamma(G) = \gamma(G_1) + \gamma(G_2)$. Then whenever A and B are γ- sets of G_1 and G_2 respectively, $A \cup B$ is a dominating set of G. So $\gamma(G) \leq |A \cup B| \leq |A| + |B| = \gamma(G_1) + \gamma(G_2)$. As $\gamma(G) = \gamma(G_1) + \gamma(G_2)$, it follows that $A \cup B$ is a γ- set of G...(2).

Clearly $u \notin A \cap B$, whenever A and B are γ- sets of G_1 and G_2 respectively. From (2) as G is just excellent, G_1 and G_2 have unique γ- sets. Atleast one of D_1 or D_2 dominates u. Assume that D_1 dominates u. Then D_1 is a γ- set of G_1. [If D_1 is not a γ- set of G_1, let A be a γ- set of G_1. Then $A \cup D_2$ is a dominating set of G and $|A \cup D_2| < |D_1 \cup D_2| = |D| = \gamma(G)$ a contradiction]. Either $|S_1| = \gamma(G_1)$ *or* $|S_2| = \gamma(G_2)$. [Otherwise $|S| = |S_1| + |S_2| - 1 \geq \gamma(G_1) + 1 + \gamma(G_2) + 1 - 1$, which is a contradiction]. Since D_1 is a γ- set of G_1 and $D_1 \neq S_1$, S_1 is not a γ-set of G_1. So S_2 is a γ- set of G_2. Now $D_1 \cup S_2$ and D are γ- sets of G, which is a contradiction as G is just excellent.

Thus, $\gamma(G) \neq \gamma(G_1) + \gamma(G_2)$ ... (3)

From the relation $|S| = \gamma(G) = \gamma(G_1) + \gamma(G_2) - 1$ and $|S_i| \geq \gamma(G_i)$ we get $|S_1| = \gamma(G_1)$ and $|S_2| = \gamma(G_2)$.

The vertex u cannot be a level vertex of both G_1 and G_2. [If not as D_i dominates $G_i - u$, $|D| = |D_1| + |D_2| = \gamma(G_1) + \gamma(G_2)$, which is a contradiction to (3)].

Also u cannot be a nonlevel vertex of both G_i. [If so select $A_i \subseteq V(G_i)$ such that A_i dominates $G_i - u$ and $|A_i| = \gamma(G_i) - 1$. Clearly $u \notin A_i$, and $A_1 \cup A_2 \cup \{w\}$ is a γ- set of G for any $w \in N[u]$ which is a contradiction as G is just excellent]. Let u be a level vertex of G_1 and a non level vertex of G_2. Let $B \subseteq V(G_2)$ such that B dominates $G_2 - u$, and $|B| = \gamma(G_2) - 1$. As $|S_1| = \gamma(G_1)$, S_1 is a γ- set of G_1. Since D_1 dominates $G_1 - u$ and u is a level vertex of G_1, $|D_1| \geq \gamma(G_1)$, $|D_2| \leq \gamma(G_2) - 1$ and D_2 is not a γ - set of G_2. Therefore D_1 dominates u and D_1 is a γ- set of G_1. Now $S_1 \cup B$ and $D_1 \cup B$ are γ- sets of G, which is a contradiction. In all cases we get a contradiction. Thus G has no cut vertex.

Theorem 5:

In a just excellent graph, $G \neq \overline{K_n}$ every vertex u is a level vertex, and also $\gamma(G - u) = \gamma(G)$.

Proof:

Let u be a vertex in G. Then there exist a γ - set of G not containing u. Hence $\gamma(G - u) \leq \gamma(G)$. We claim that $\gamma(G - u) = \gamma(G)$. If possible assume that $\gamma(G - u) < \gamma(G)$. So $G \neq K_n$. Let S be a γ - set for $G - u$. Then $S \cup \{ v \}$ is a γ - set for G, $\forall$ $v \in N[u]$. As, G is connected, $N[u]$ contains atleast two vertices. So $S \cup \{ u \}$ and $S \cup \{ v \}$ are γ - sets for G for all $v \in N(u)$, a contradiction as G is just excellent. So $\gamma(G - u) = \gamma(G)$. If $\gamma^u(G, u) < \gamma(G)$, let S be a $\gamma^u(G, u)$ - set . If $u \in S$, then S is also a dominating set for G, which is a contradiction. If $u \notin S$, then S is a γ - set for $G - u$ and $\gamma(G - u) < \gamma(G)$ which is also a contradiction. Thus, $\gamma^u(G, u) = \gamma(G)$.

The converse of the above theorem need not be true. For example in C_{3n+2}, every vertex is a level vertex, but C_{3n+2} is not just excellent.

Theorem 6:

Let S_1, S_2, S_3 be the distinct γ - sets of C_{3n}. Then

1. *If $A \subset (E(\overline{C_{3n}})$ such that for every edge $e \in A$, e has both the end vertices in S_1, then $G = C_{3n} + A$ is just excellent. Further $d (G) = 3$.*
2. *If $u \in S_i$ and $v \in S_j$, $j \neq i$, then $C_{3n} + uv$ is not just excellent, where $uv \notin E (C_n)$.*

Proof:

Let $\{v_0, v_2, ..., v_{3n-1}\}$ be the vertices taken in clockwise order on the cycle C_{3n}. Without loss of generality let $S_i = \{ v_{3k+i} \mid k = 0, 1, ..., n - 1 \}$ where $i = 1, 2, 3$ (under addition modulo $3n$). Let D be a γ - set of G. If $D \cap S_1 = \phi$, then D is a γ - set of C_{3n} also. So in this case $\gamma(G) = \gamma(C_{3n}) = n$ and $D = S_2$ or S_3. If $D \cap S_1 \neq \phi$, we claim that $D = S_1$.

Assume that $v_{3t} \in D$, but $v_{3(t+1)} \notin D$. D must contain atleast one vertex from each of the subsets $\{ v_j \mid j = 3t + 1, 3t + 2 \}$, $\{ u_j \mid j = 3(t + k), 3(t + k) + 1, 3(t + k) + 2 \}$, where $k = 1, 2, ..., n - 1$ in order to dominate the vertices $v_{3(t+k)+1}$ and v_{3t+2}. As D also contains v_{3t}, $| D | > n$ a contradiction as $\gamma(G) \leq \gamma(C_{3n}) = n$. So $v_{3t} \in D \Rightarrow v_{3(t+1)} \in D$ and hence $S_1 \subseteq D$. As $\gamma(G) \leq \gamma(C_{3n})$, $D = S_1$. Then S_1, S_2, S_3 are the only γ - sets of G and hence G is just excellent and $d (G) = 3$.

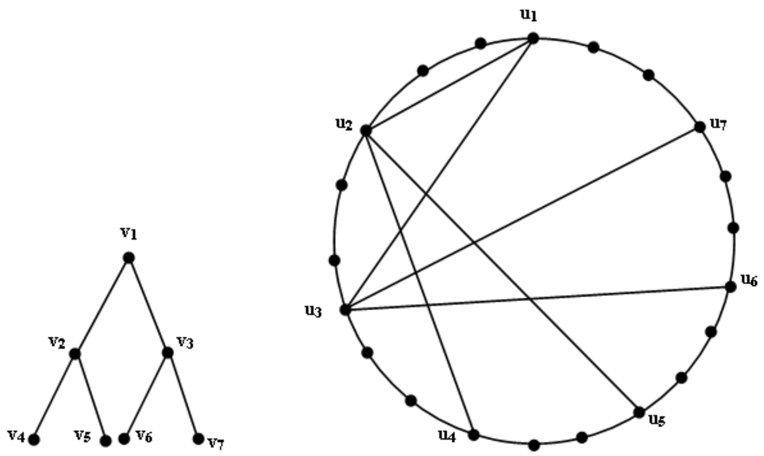
Let $u \in S_i$ and $v \in S_j$, i $\neq j$ and $uv \notin E$ (C_n). Let $j = i + 1$(*mod 3*). Let $u = v_{3t+1}$ and v = v_{3t+j}. Then S_j and ($S_j - v_{3t+j}$) $\cup$ { v_{3t+i+2} } are γ- sets of $G = C_n + uv$. Hence *G* is not just excellent.

Theorem 7:

Every graph is an induced subgraph of a just excellent graph.

Proof:

Let *G* be the given graph. If *G* is just excellent, then there is nothing to prove. Assume that *G* is not just excellent. Let V (G) = { $v_1, v_2, ..., v_n$ }. Consider the cycle C_{3n}. It is just excellent. Let S_1, S_2, S_3 be the distinct γ- sets of C_{3n}. Label the vertices of S_1 by u_1, u_2, ... , u_n. Now in C_{3n} we add edges u_iu_j if and only if v_iv_j is an edge in *G*. Let the resulting graph be *H*. Then the induced subgraph $< S_1 >$ in *H* is isomorphic to *G*. By theorem 3, *H* is just excellent and γ(*H*) = *n*. Thus, the given graph *G* is an induced subgraph of a just excellent graph *H*. For example consider the tree in Fig. 2(a). It is not just excellent. This tree can be imbedded in a just excellent graph G as seen in Fig. 2(b).



(a) Fig. 2 (b)

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