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Eccentric Domatic Number of a Graph

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Abstract: *A subset D of the vertex set V(G) of a graph G is said to be a dominating set if every vertex not in D is adjacent to at least one vertex in D. A dominating set D is said to be an eccentric dominating set if for every $v \in V-D$, there exists at least one eccentric point of v in D. The minimum of the cardinalities of the eccentric dominating sets of G is called the eccentric domination number* $\gamma_{ed}(G)$ *of G. A partition of V(G) is called eccentric domatic if all its classes are eccentric dominating sets in G. The maximum number of classes of an eccentric domatic partition of V(G) is called the eccentric domatic number of G and is denoted by* $d_{ed}(G)$*. In this paper, bounds for* $d_{ed}(G)$ *and its exact value for some particular classes of graphs are studied.*

Key words: *Eccentric dominating set, Eccentric domination number, Eccentric domatic number.*

1.Introduction

Let G be a finite, simple, undirected graph on n vertices with vertex set V(G) and edge set E(G). For graph theoretic terminology and domination concepts refer to Harary [4], Buckley and Harary [1] and Haynes, Hedetniemi, and Slater [8].

Definition 1.1 Let G be a connected graph and u be a vertex of G. The **eccentricity** e(v) of v is the distance to a vertex farthest from v. Thus, $e(v) = \max \{d(u, v) : u \in V\}$. The **radius** r(G) is the minimum eccentricity of the vertices, whereas the **diameter** diam(G) is the maximum eccentricity. For any connected graph G, $r(G) \leq diam(G) \leq 2r(G)$. v is a central vertex if e(v) = r(G). The **center C(G)** is the set of all central vertices. The central subgraph < C(G) > of a graph G is the subgraph induced by the center. v is a peripheral vertex if e(v) = d(G). The periphery P(G) is the set of all peripheral vertices.

For a vertex v, each vertex at a distance e(v) from v is an **eccentric vertex of v**. The set of all eccentric vertices of v is known as the **eccentric set E(v) of v**.

Definition 1.2 The **open neighborhood** N(u) of a vertex v is the set of all vertices adjacent to v in G. $N[v] = N(v) \cup \{v\}$ is called the **closed neighborhood** of v. For a vertex $v \in V(G)$, $N_i(u) = \{u \in V(G) : d(u, v) = i\}$ is defined to be the i^{th} **neighborhood** of v in G.

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Definition 1.3 [2,8] A set $S \subseteq V$ is said to be a **dominating set** in G, if every vertex in $V-S$ is adjacent to some vertex in S.

Definition 1.4 [3] A partition of V(G) is called **domatic** if all of its classes are dominating sets in G. The maximum number of classes of an domatic partition of V(G) is called the **domatic number** of G and is denoted by $d_d(G)$.

Definition: 1.5 [6] A set $D \subseteq V(G)$ is an **eccentric dominating set** if D is a dominating set of G and for every $v \in V - D$, there exists at least one eccentric point of v in D. If D is an eccentric dominating set, then every superset $D' \supseteq D$ is also an eccentric dominating set. But $D'' \subseteq D$ is not necessarily an eccentric dominating set.

An eccentric dominating set D is a **minimal eccentric dominating set** if no proper subset $D'' \subseteq D$ is an eccentric dominating set.

Definition: 1.6 [6] The **eccentric domination number $\gamma_{ed}(G)$** of a graph G equals the minimum cardinality of an eccentric dominating set. That is, $\gamma_{ed}(G) = \min |D|$, where the minimum is taken over D in D, where D is the set of all minimal eccentric dominating sets of G.

Obviously, $\gamma(G) \leq \gamma_{ed}(G)$.

Partitioning a given graph into sets such that each of which has some specified property has many applications in clustering a communication network. So in this section we define a new parameter known as **eccentric domatic number** of a given graph and study that parameter.

Definition: 1.6 A partition of V(G) is called **eccentric domatic** if all of its classes are eccentric dominating sets in G. The maximum number of classes of an eccentric domatic partition of V(G) is called the **eccentric domatic number of G** and is denoted by $d_{ed}(G)$.

For every graph G, there exists at least one eccentric partition of V(G), namely {V(G)}. Therefore, $d_{ed}(G)$ is well defined for every graph G. We give results on this parameter in section 2.

In [6], we have established the following results and are needed to study the eccentric domatic number of some classes of graphs.

Theorem:1.1 [6] $\gamma_{ed}(K_{,n}) = 1$.

Theorem:1.2 [6] If G is of diameter two $\gamma_{ed}(G) \leq 1+\delta(G)$.

Theorem: 1.3 [6] $\gamma_{ed}(P_n) = \lceil n/3 \rceil$, if n = 3k+1,

$$\gamma_{ed}(P_n) = \lceil n/3 \rceil + 1, \text{ if } n = 3k \text{ or } 3k+2.$$

Theorem: 1.4[6] (i) $\gamma_{ed}(C_n) = n/2$ if n is even.

(ii) $\gamma_{ed}(C_n) = \lceil n/3 \rceil$ or $\lceil n/3 \rceil + 1$, if n is odd.

Theorem: 1.5[6] $\gamma_{ed}(\overline{C_4}) = 2$, $\gamma_{ed}(\overline{C_5}) = 3$ and $\gamma_{ed}(\overline{C_n}) = \lceil n/3 \rceil$, $n \geq 6$.

2. Eccentric domatic number of a graph

First, we shall give some bounds for the eccentric domatic number of a graph.

Observation:2.1 If D is an eccentric dominating set in G, then for each $v \in V - D$, $D \cap N(v)$ and $D \cap E(v)$ are non empty sets. Hence we have, $d_{ed}(G) \leq 1 + \delta(G)$ and $d_{ed}(G) \leq \min_{v \in V(G)} \left(1 + |E(v)|\right)$.

Observation:2.2 If G is a graph on n vertices, then $1 \leq d_{ed}(G) \leq n$.

Now we give some observations, theorems and propositions relating eccentric domatic numbers of some classes of graphs.

Observation:2.3 For $n \geq 2$, $d_{ed}(K_{1,n}) = 1$.

Observation:2.4 For $n \geq 3$, $d_{ed}(P_n) = 1$.

Observation: 2.5 If G is a tree, $d_{ed}(G) \leq 2$.

Since every vertex of K_n is an eccentric dominating set the following proposition holds.

Proposition: 2.1 $d_{ed}(K_n) = n$.

In observation 2.2, both lower and upper bounds are sharp since $d_{ed}(K_n) = n$ and $d_{ed}(K_{1,n}) = 1$.

$d_{ed}(K_{m,n})$ is given by the following proposition.

Proposition: 2.2 For $2 \leq m \leq n$, $d_{ed}(K_{m,n}) = m$.

Proof: Let V_1, V_2 be the bipartition classes of $K_{m,n}$. Consider $u \in V_1$, $v \in V_2$. Clearly, {u,v} is a γ_{ed} dominating set.

Let $V_1 = \{u_1, u_2, \ldots, u_m\}$, $V_2 = \{v_1, v_2, \ldots, v_n\}$. Then $\{u_i, v_i\}$, i = 1, 2, 3,…, m-1, $\{u_m, v_m, v_{m+1}, \ldots, v_n\}$ form an eccentric domatic partition of $K_{m,n}$. Hence $d_{ed}(K_{m,n}) \geq m$. On the other hand, $d_{ed}(K_{m,n}) \leq d(K_{m,n}) = m$. Therefore, $d_{ed}(K_{m,n}) = m$.

Proposition: 2.3 If $d_{ed}(G) \geq 3$, then $\delta(G) \geq 2$.

Proof: If $d_{ed}(G) \geq 3$, then every vertex of G has atleast two neighbours. Thus $\delta(G) \geq 2$.

Now, we will prove some results related to unique eccentric point graphs.

If G is an unique eccentric point graph, no vertex of G has more than one eccentric vertex hence we have

Proposition: 2.4 If G is an unique eccentric point graph then $d_{ed}(G) \leq 2$.

Proposition: 2.5 If G is an unique eccentric point graph such that each vertex is an eccentric point of a unique vertex with odd number of vertices then $d_{ed}(G) = 1$.

Proof: If G is an unique eccentric point graph, $\gamma_{ed}(G) \geq n/2$. Also G has an odd number of vertices. Hence $d_{ed}(G) = 1$.

Theorem 2.1 If G is an unique eccentric point graph with $\delta(G) = 1$, then $d_{ed}(G) = 1$.

Proof: If G is an unique eccentric point graph then $d_{ed}(G) \leq 2$ by Proposition 2.5. Let $u \in V(G)$ such that deg u = 1, and let v be the support of u in G. Since u is pendent, u and v have the same eccentric point w (unique).

Now, suppose that $d_{ed}(G) = 2$. Let $\{D_1, D_2\}$ be an eccentric domatic partition of G and let $u \in D_1$. Then $v \in D_2$ and $w \in D_2$. But $v \in D_2$ and D_1 is an eccentric dominating set implies that $w \in D_1$, which is a contradiction. Hence $d_{ed}(G) = 1$.

Theorem: 2.2 Let n be an even positive integer. Let G be obtained from the complete graph K_n by deleting edges of linear factor, then $d_{ed}(G) = 2$.

Proof: Let u and v be a pair of non-adjacent vertices in G. Then u and v are eccentric to each other. Also, G is a unique eccentric point graph and each vertex is an eccentric point of exactly one vertex. Therefore, $\gamma_{ed}(G) \geq n/2$. G is also regular self-centered with diameter 2. Consider, $D \subseteq V(G)$ such that $\langle D \rangle = K_{n/2}$. D contains n/2 vertices each vertex in V – D is adjacent to atleast one element in D and each element in V – D has its eccentric point in D. Hence $\gamma_{ed}(G) = n/2$. Also, there exists only two such partitions. Hence, $d_{ed}(G) = 2$.

In the following two theorems we study the number of domatic partitions of C_n and its complement.

Theorem: 2.3 (i) $d_{ed}(C_n) = 2$, if n is even.

(ii) $d_{ed}(C_n) = 2$, if n is odd and $n \neq 3m$.

(iii) $d_{ed}(C_n) = 3$, if $n = 3m$ and is odd.

Proof of (i): Let the cycle C_n be $v_1 v_2 v_3 \ldots v_{2k} v_1$. Each vertex of C_n has exactly one eccentric vertex and $\gamma_{ed}(C_n) = n/2$ if n is even. Hence $d_{ed}(C_n) \leq 2$.

If n = 4, any two adjacent vertices of C_4 is an eccentric dominating set of C_4.

Hence $d_{ed}(C_4) = 2$.

Let n = 2k and k > 2.

case(i) k-odd.

Consider $D_1 = \{v_1, v_3, \ldots, v_k, v_{k+2}, \ldots, v_{2k-1}\}$ and $D_2 = \{v_2, v_4, \ldots, v_{k-1}, v_{k+1}, \ldots, v_{2k}\}$. This D_1 and D_2 is are eccentric dominating sets for C_n since they dominates C_n and v_i is an eccentric point of v_{i+k}. $\{D_1, D_2\}$ is an eccentric domatic partition of G. Hence $d_{ed}(C_n) = 2$.

case(ii) k even.

Let $D_1 = \{v_1, v_3, \ldots, v_{k-1}, v_{k+2}, v_{k+4}, \ldots, v_{2k}\}$. $D_2 = \{v_2, v_4, \ldots, v_k, v_{k+1}, v_{k+3}, \ldots, v_{2k-1}\}$. This D_1 is an eccentric dominating set for C_n since D_1 dominates C_n and v_i is an eccentric point of v_{i+k}. $\{D_1, D_2\}$ is an eccentric domatic partition of G. Hence $d_{ed}(C_n) = 2$.

Proof of (ii): Each vertex of C_n has exactly two eccentric vertices and two adjacent vertices and $\gamma_{ed}(C_n) = \lceil n/3 \rceil$ or $\lceil n/3 \rceil + 1$, if n is odd. Therefore, $d_{ed}(C_n) \leq 3$.

If $n = 2k+1$, $v_i \in V(G)$ has v_{i+k}, v_{i+k+1} as eccentric points.

case(i) $n = 3m+1$, n odd $\Rightarrow$ m is even.

Also $3m = 2k \Rightarrow$ k is a multiple of 3.

Consider $D = \{v_1, v_4, \ldots, v_{k+1}, v_{k+3}, v_{k+6}, \ldots, v_{2k-1}\}$. D is an eccentric dominating set and $|D| = \lceil n/3 \rceil = \gamma(C_n)$. Hence $\gamma_{ed}(C_n) = \lceil n/3 \rceil = m+1$. So $d_{ed}(C_n) \leq 2$. $V-D$ is also an eccentric dominating set. Therefore, $d_{ed}(C_n) = 2$.

case(ii) $n = 3m+2 \Rightarrow 3m$ is odd $\Rightarrow$ m is odd.

$$2k = 3m+1 = 3(m-1) + 4$$

$$k = 3l + 2$$

Consider $D = \{v_1, v_4, \ldots, v_{k-1}, v_k, v_{k+3}, \ldots, v_{2k+1}\}$. D is an eccentric dominating set with $\lceil n/3 \rceil + 1$ vertices and no γ-dominating set of C_n is an eccentric dominating set of C_n.

Hence $\gamma_{ed}(C_n) = \lceil n/3 \rceil + 1$ and as in the previous case $d_{ed}(C_n) = 2$.

Proof of (iii): When $n = 3m$, $\gamma_{ed}(C_n) = n/3 = \gamma(C_n)$. Each vertex of C_n has exactly two eccentric vertices and two adjacent vertices. Therefore, $d_{ed}(C_n) \leq 3$.

$n = 3m$, n odd $\Rightarrow$ m odd

$n = 3m = 2k+1 \Rightarrow 2k$ even and $2k = 3m-1$

$$2k = 3(m-1)+2$$

$k = (3(m-1)+2)/2 \Rightarrow k = 3l+1$ (since $m-1$ is even)

Consider $D_1 = \{v_1, v_4, v_7, \ldots, v_k, v_{k+3}, v_{k+6}, \ldots, v_{2k-1}\}$,

$D_2 = \{v_2, v_5, v_8, \ldots, v_{k+1}, v_{k+4}, v_{k+7}, \ldots, v_{2k}\}$,

$D_3 = \{v_3, v_6, v_9, \ldots, v_{k+2}, v_{k+5}, v_{k+8}, \ldots, v_{2k+1}\}$.

Then D_1, D_2, D_3 form an eccentric domatic partition of $V(C_n)$. Hence, $d_{ed}(C_n) = 3$.

Theorem: 2.4 $d_{ed}(\overline{C_n}) = 2$ if $n \neq 3m$,

$d_{ed}(\overline{C_n}) = 3$ if $n = 3m$,

Proof: We know, $\gamma_{ed}(\overline{C_4}) = 2$, $\gamma_{ed}(\overline{C_5}) = 3$ and $\gamma_{ed}(\overline{C_n}) = \lceil n/3 \rceil$, $n \geq 6$ and each vertex of $\overline{C_n}$ has exactly two eccentric vertices when $n > 3$. Therefore, $d_{ed}(\overline{C_n}) \leq 3$.

Case 1: $n = 3m$

Let us assume that $v_1, v_2, \ldots, v_n, v_1$ form C_n. Then $D_1 = \{v_1, v_4, \ldots, v_{3m-2}\}$; $D_2 = \{v_2, v_5, \ldots, v_{3m-1}\}$; $D_3 = \{v_3, v_6, \ldots, v_{3m}\}$ form a partition of $V(\overline{C_n})$ into minimum eccentric dominating sets of $\overline{C_n}$. Hence $d_{ed}(\overline{C_n}) = 3$.

Case 2: $n \neq 3m$

In this case, $d_{ed}(\overline{C_n}) \leq 2$ since $\gamma_{ed}(\overline{C_n}) = \lceil n/3 \rceil > n/3$.

If $n = 3m+1$, $D = \{v_1, v_4, \ldots, v_{3m+1}\}$ and $V-D$ form an eccentric domatic partition of $\overline{C_n}$. Hence $d_{ed}(\overline{C_n}) = 2$.

If $n = 3m+2$, $D = \{v_1, v_4, \ldots, v_{3m+1}, v_{3m+2}\}$ and $V-D$ form an eccentric domatic partition of $\overline{C_n}$. Hence $d_{ed}(\overline{C_n}) = 2$.

Next theorem gives the eccentric domatic number of wheels.

Theorem: 2.5 $d_{ed}(W_3) = 4$, $d_{ed}(W_4) = 2$, $d_{ed}(W_5) = 2$, $d_{ed}(W_6) = 3$, $d_{ed}(W_7) = 2$ and $d_{ed}(W_n) = 3$, $n \geq 8$.

Proof: $W_3 \cong K_4$. Hence, $d_{ed}(W_3) = 4$.

$W_4 = K_1+C_4$. Let v_1, v_2, v_3, v_4 be the vertices of C_4 and v be the central vertex of W_4. Then $\{v, v_1, v_2\}$, $\{v_3, v_4\}$ are eccentric partitions of W_4. Hence $d_{ed}(W_4) = 2$.

Similarly, we can prove that $d_{ed}(W_5) = 2$, $d_{ed}(W_6) = 3$ and $d_{ed}(W_7) = 2$.

When $n \geq 8$, $W_n = K_1+C_n$.

Let $v_1, v_2, \ldots, v_n$ be the n vertices of C_n, then the central vertex v with any two vertices of C_n at distance three or more in C_n or v with any two adjacent vertices of C_n form an eccentric dominating set of W_n. Also, any dominating set of C_n is an eccentric dominating set of W_n. Hence, $d_{ed}(W_n) = d(C_n) = 3$ for $n \geq 8$.

A lower bound for eccentric domatic number is given in the following theorems.

Theorem: 2.6 If G is of radius greater than two, then $d_{ed}(G) \geq \lfloor n/(n-\delta(G) \rfloor$.

Proof: If G is a graph with radius greater than two, V − N(u), where deg u = $\delta(G)$ is an eccentric dominating set. Let $D \subseteq V(G)$ with $|D| \geq n-\delta(G)$, then D is an eccentric dominating set. Thus we can take any $\lfloor n/(n-\delta(G)\rfloor$ disjoint subsets. Hence, $d_{ed}(G) \geq \lfloor n/(n-\delta(G)\rfloor$.

Theorem: 2.7 If G is self-centered of radius two, then $d_{ed}(G) \geq \lfloor n/(1+\Delta(G))\rfloor$.

Proof: If G is self-centered of radius two, N[u] is an eccentric dominating set for any $u \in V(G)$. Let $D \subseteq V(G)$ with $|D| \geq 1+\Delta(G)$, then D is an eccentric dominating set. Thus we can take any $\lfloor n/(1+\Delta(G))\rfloor$ disjoint subsets as eccentric dominating sets. Hence, $d_{ed}(G) \geq \lfloor n/(1+\Delta(G))\rfloor$.

Nordhaus-Gaddum type of results involving the domatic number of G and its complement is given in the following theorem.

Theorem: 2.8 $d_{ed}(G) + d_{ed}(\overline{G}) \leq n+1$ with equality if and only if $G = K_n$ or $\overline{K_n}$.

Proof: $d_{ed}(G) = 1$ and $d_{ed}(\overline{G}) = n$ when $G = K_n$ or $\overline{K_n}$. If $G \neq K_n$ or $\overline{K_n}$, $\gamma_{ed}(G) \geq 2$. Therefore $d_{ed}(G) \leq \lfloor n/2\rfloor$. Thus we see that $d_{ed}(G) + d_{ed}(\overline{G}) \leq n$.

Next, we proceed to prove eccentric domatic number of a graph or its complement is greater than two when the radius of G is greater than two.

Theorem: 2.9 Let G be a connected graph with radius greater than two. Then there exists a minimal eccentric dominating set D of G with the property that for $u \in D$, $N(u) \not\subset D$ and $V - N(u) \not\subset D$.

Proof: Let D be any minimal eccentric dominating set of G. For any $u \in V(G)$, V − N(u) is an eccentric dominating set of G, but it is not minimal by theorem 2.5.. Hence $V - N(u) \not\subset D$. Next we prove that there exists D such that $N(u) \not\subset D$. Since G is of radius greater than two $D \neq N[u]$, since N[u] is not a dominating set of G and D must contain vertices which are at distance atleast two. Since D is minimal, N(u) is a subset of D if and only if each vertex of N(u) is the support of some pendent vertices. Let S_u be the set of all such pendent vertices. Take $D' = (D - N(u))\cup S_u$. D' is a minimal eccentric dominating set of G (if x is a pendent vertex and y its support then x and y have same eccentric vertex) and $N(u) \not\subset D'$. Hence we can form a minimal dominating set D such that $N(u) \not\subset D$ for any $u \in D$. This proves the theorem.

Theorem: 2.10 Let G be a connected graph with radius $r > 2$. Then either G or $\overline{G}$ has atleast two disjoint eccentric dominating sets; that is $d_{ed}(G)$ or $d_{ed}(\overline{G})$ is ≥ 2.

Proof: When radius of G is greater than two, $\overline{G}$ is self-centered of diameter two. Hence in $\overline{G}$ two vertices are at distance two to each other implies that they are eccentric to each other. Hence for any $u \in V(G)$, $V-N(u)$ is an eccentric dominating set. Since radius of G is greater than two, vertices in N(u) has their eccentric vertices atleast at distance two from u. So, we can leave atleast one vertex v from $N_2(u)$ or from $N_3(u)$ such that $(V - N(u)) - \{v\}$ form an eccentric dominating set of G. Hence, $\gamma_{ed}(G) < n - \Delta(G)$. Thus we can have a minimal dominating set $D \subseteq V-N(u)$ of G. Let us prove that D and $V-D$ are eccentric dominating sets of $\overline{G}$.

Let $v \in V-D$. Since D is an eccentric dominating set of G, there exists $u, w \in D$ in G such that u is adjacent to v and w is eccentric to v in G. Therefore in $\overline{G}$, u is eccentric to v and w is adjacent to v. This proves that D is an eccentric dominating set of $\overline{G}$.

Now take $u \in D$. In G, u has some adjacent vertices in $V-D$ and some non-adjacent vertices in $V-D$. Hence in $\overline{G}$, u has some adjacent vertices in $V-D$ and some non-adjacent vertices that is eccentric vertices in $V-D$. This implies $V-D$ is also an eccentric dominating set of $\overline{G}$. Hence, $\overline{G}$ has atleast two disjoint eccentric dominating sets namely D and $V-D$. Therefore, $d_{ed}(\overline{G}) \geq 2$.

Corollary: 2.10 Let T be a tree with radius $r > 2$. Then $d_{ed}(\overline{T}) = 2$.

Theorem: 2.11 If G is a connected graph with radius > 2, then $2 \leq d_{ed}(G)\, d_{ed}(\overline{G}) \leq n^2/4$.

Proof: As in theorem 2.9, $d_{ed}(\overline{G}) \geq 2$. Thus, $2 \leq d_{ed}(G)\, d_{ed}(\overline{G}$ Now by theorem 2.7 $d_{ed}(G) + d_{ed}(\overline{G}) \leq n$. Thus $\sqrt{d_{ed}(G)\, d_{ed}(\overline{G})} \leq (d_{ed}(G) + d_{ed}(\overline{G}))/2 \leq n/2$; That is $d_{ed}(G)\, d_{ed}(\overline{G}) \leq n^2/4$.

Following theorem sharpened the bounds of theorem 2.8 for trees.

Theorem: 2.12 For any tree of order $n \geq 2$, $2 \leq d_{ed}(T) + d_{ed}(\overline{T}) \leq 4$. If radius of T is greater than two, then $3 \leq d_{ed}(T) + d_{ed}(\overline{T}) \leq 4$.

Proof: We know that $d_{ed}(T) \leq 2$. Also, since an end vertex of T has exactly one eccentric vertex in $\overline{T}$, $d_{ed}(\overline{T}) \leq 2$. Thus, $2 \leq d_{ed}(T) + d_{ed}(\overline{T}) \leq 4$. When the radius is greater than two, by theorem 2.9, $d_{ed}(\overline{T}) \geq 2$. So $d_{ed}(\overline{T}) = 2$. Thus we get $3 \leq d_{ed}(T) + d_{ed}(\overline{T}) \leq 4$.

The bounds in the previous theorem are sharp. When r = 2, $d_{ed}(T) + d_{ed}(\overline{T})=2$ for $T = P_4$. $d_{ed}(T) + d_{ed}(\overline{T}) = 4$ for a spider having more than three legs.

$d_{ed}(T) + d_{ed}(\overline{T})=3$ if $T = P_7$. $d_{ed}(T) + d_{ed}(\overline{T}) = 4$ if T is a tree as in figure 2.1.

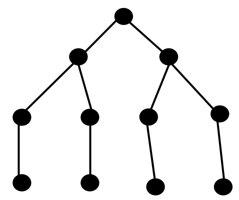

Figure 2.1

In the next theorem, we characterize trees T for which $d_{ed}(T) = 2$

Theorem: 2.13 Let T be a tree with diameter d, then $d_{ed}(T) = 2$ if and only if T satisfies the following conditions:

(i) P(T) has at least two pairs of peripheral vertices at distance d to each other.

(ii) Suppose (x, y), (z, w) are two such pairs, then support of x is different from supports of z and w and support of y is different from support of z and w.

Proof: Clearly $d_{ed}(T) \leq 2$.

Suppose T satisfies conditions (i) and (ii), let T_1 be the new graph obtained from T by removing the supports of x, y, z and w. Clearly T_1 is a tree and it has two disjoint dominating sets. Let them be D_1 and D_2. Consider $D_1' = D_1 \cup$ {supports of z and w} $\cup$ {end vertices adjacent to supports of x and y}. Clearly, x, y $\in D_1'$. Consider $D_2' = D_2 \cup$ {supports of x and y} $\cup$ {end vertices adjacent to supports of z and w}. Clearly, z, w $\in D_2'$. Then D_1', D_2' form an eccentric domatic partition of V(T). Thus $d_{ed}(T) = 2$.

On the other hand, assume that $d_{ed}(T) = 2$. Let $V(G) = D_1 \cup D_2$, where $\{D_1, D_2\}$ is an eccentric domatic partition of T. D_1 contains atleast two peripheral vertices at distance d to each other. Let them be x, y. D_2 contains atleast two peripheral vertices at distance d to each other. Let them be z, w. Therefore, T satisfies (i).

We know that x, y, z, w are end vertices of T. Since $\{D_1, D_2\}$ is an eccentric domatic partition of T, supports of x, y are in D_2 and supports of z, w are in D_1. Thus supports of x, y cannot be same as supports of z, w. This proves (ii). Hence the theorem is proved.

Now let us define eccentric domatically full graphs.

Definition 2.1 A graph G is **eccentric domatically full** if $d_{ed}(G) = 1+\delta(G)$.

Using this definition the previous theorem can be restated as follows.

Theorem: 2.14 Let T be a tree with diameter d. Then T is eccentric domatically full if and only if T satisfies the following conditions:

(i) P(T) has at least two pairs of peripheral vertices at distance d to each other.

(ii) Suppose (x, y), (z, w) are two such pairs, then support of x is different from supports of z and w and support of y is different from support of z and w.

Definition 2.2 A graph G is **eccentric domatic eccentrically full** if $d_{ed}(G) = \min_{v\in V(G)} \left(1 + |E(v)|\right)$.

Since for a tree, $\min_{v\in V(G)} |E(v)| = 1$, Theorem 2.13 characterizes trees which are eccentric domatic eccentrically full.

If T is a tree with radius r > 2 then from Corollary 2.10 it follows that $\overline{T}$ is always eccentric domatic eccentrically full

Also, a cycle C_n is eccentric domatically full if and only if n = 3m and n is odd since $d_{ed}(C_n) = 3$, if n = 3m and n is odd. C_n is eccentric domatic eccentrically full if and only if n = 3m and n is odd or n is even, since $d_{ed}(C_n) = 3$, when n = 3m and odd; and $d_{ed}(C_n) = 2$, when n is even.

Definition 2.2 A graph G is **domatically and eccentrically full** if $d_{ed}(G) = 1+\delta(G) = \min_{v\in V(G)} \left(1 + |E(v)|\right)$.

Thus trees satisfying (i) and (ii) in Theorem 2.13 are domatically and eccentrically full.

In the end, we prove an existing theorem.

Theorem: 2.15 Let V be a finite set with more than three vertices, and let k be any integer such that $1 \le k \le |V|/2$ and let $\{D_1, D_2, ..., D_k\}$ be a partition of V with $|D_i| \ge 2$. Then there exists a self-centered graph G with V(G) = V and $\{D_1, D_2, ..., D_k\}$ as an eccentric domatic partition.

Proof: In each D_i taking the elements as vertices, join each vertex to all other vertices by edges. Therefore, $\langle D_i \rangle$ is a complete graph for all i. Now, for any two distinct D_i, D_j split D_i into two parts X_{1i}, X_{2i} and D_j into two parts Y_{1j}, Y_{2j}. Join each vertex of X_{1i} to all the vertices of Y_{1j} and each vertex of X_{2i} to all vertices of Y_{2j}. But no vertex of X_{1i} is joined to vertices of Y_{2j} and no vertex of X_{2i} is joined to vertices of Y_{1j}. Name the new graph formed as G. Clearly G is self-centered of diameter 2. Also, for any i, D_i is an eccentric dominating set of G. Hence, $\{D_1, D_2, ..., D_k\}$ is an eccentric domatic partition of G and $d_{ed}(G) \ge k$. If $|D_i| = 2$ for all i, $d_{ed}(G)$ is exactly k since $\gamma_{ed}(G) \ge 2$ for $G \ne K_n$.

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