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Distance Closed Domination in Graph

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Abstract: *In a graph G=(V,E), a set $S \subset V(G)$ is a distance closed set of G if for each vertex $u \in S$ and for each $w \in V\text{-}S$, there exists at least one vertex $v \in S$ such that $d_{<S>}(u, v) = d_G(u, w)$. Also, a vertex subset D of V(G) is a dominating set of G if each vertex in V-D is adjacent to at least one vertex in D. In this paper, we define a new concept of domination called distance closed domination (D.C.D) and analyze some structural properties of graphs and extremal problems relating to the above concepts.*

Keywords: *domination number, distance, eccentricity, radius, diameter, self centered graph, neighborhood, induced sub graph, unique eccentric point graph, ciliates, distance closed dominating set.*

1. Introduction

Graphs discussed in this paper are undirected and simple. Unless otherwise stated the graphs which we consider are connected and simple graphs only. For a graph, let V(G) and E(G) denotes its vertex and edge set respectively. The *degree* of a vertex v in a graph G is denoted by $deg_G(v)$. The length of any shortest path between any two vertices u and v of a connected graph G is called the *distance between u and v* and it is denoted by $d_G(u, v)$. The distance between two vertices in different components of a disconnected graph is defined to be ∞. For a connected graph G, the *eccentricity* $e_G(v) = \max \{d_G(u,v): \forall u \in V(G)\}$. If there is no confusion, we simply use the notion deg (v), d(u, v) and e(v) to denote degree, distance and eccentricity respectively for the connected graph. The minimum and maximum eccentricities are the *radius* and *diameter* of G, denoted by r(G) and diam (G) respectively. If these two are equal in a graph, that graph is called *self-centered* graph with radius r and is called an *r-self centered* graph. Such graphs are 2-connected graphs. Some structural properties are studied in [2] and [3]. A vertex u is said to be an eccentric vertex of v in a graph G, if d (u, v)=e(v) in that graph. In general, u is called an *eccentric vertex*, if it is an eccentric vertex of some vertex. For $v \in V(G)$, the *neighborhood* $N_G(v)$ of v is the set of all vertices adjacent to v in G. The set $N_G[v] = N_G(v) \cup \{v\}$ is called the *closed neighborhood* of v. A set S of edges in a graph is said to be independent if no two of the edges in S are adjacent. An edge e=(u, v) is a *dominating edge* in a graph G if every vertex of G is adjacent to at least one of u and v.

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The concept of distance in graph plays a dominant role in the study of structural properties of graphs in various angles using related concept of eccentricity of vertices in graphs. The study of structural properties of graphs using distance and eccentricity started with the study of centre of tree and propagated in different directions in the study of structural properties of graphs such as unique eccentric point graphs, k- eccentric point graphs, self centered graphs, graphs realizing given eccentricity sequence, radius, diameter and eccentric critical graphs and Hamiltonian properties in iterated line graphs. The structural and eccentricity properties of various graph operations and iterated graph operations are given in references [4], [5], [8], [10], [12] and [14].

The concept of domination in graphs was introduced by Ore [13]. A set $D \subseteq V(G)$ is called dominating set of G if every vertex in V(G)-D is adjacent to some vertex in D. D is said to be a minimal dominating set if D-{v} is not a dominating set for any $v \in D$. The *domination number* γ(G) of G is the minimum cardinality of dominating sets. We call a set of vertices a γ-set if it is a dominating set with cardinality γ(G).Different types of dominating sets have been studied by imposing conditions on the dominating sets. The sub graph of a graph G whose vertex set is S and whose edge set is the set of those edges of G that have both ends in S is called the induced sub graph of G induced by S and is denoted by <S>. A dominating set D is called *connected* (*independent*) dominating set if the induced sub graph <D> is connected (independent). D is called a *total dominating set* if every vertex in V(G) is adjacent to some vertex in D. One important aspect of the concept of distance and eccentricity is the existence of polynomial time algorithm to analyze them. The concept of domination in graphs originated from the chess games theory and that paved the way to the development of the study of various domination parameters and its relation to various other graph parameters. The list of survey of domination theory papers are in [6], [7], [11], [15], [16] and [17].

The new concepts such as distance closed sets, distance preserving sub graphs, eccentricity preserving sub graphs, Super eccentric graph of a graph, pseudo geodetic graphs are introduced and structural properties of those graphs are studied in [9].

Janakiraman and Alphonse [1] introduced and studied the concept of weak convex dominating sets, which mixes the concept of dominating set and distance preserving set. In this paper, we introduce a new dominating set called distance closed dominating set of a graph through which we studied the properties of the graph. We find upper and lower bounds for the new domination number in terms of various already known parameters. Also, we studied several interesting properties like Nordhaus-Gaddum type results relating the graph and its complement.

2. Prior Results

The concept of distance closed set is defined and studied in the doctoral thesis of Janakiraman [9] and the concept of distance closed sets in graph theory is due to the related concept of ideals in ring theory in algebra. The ideals in a ring are defined with respect to the multiplicative closure property with the elements of that ring. Similarly, the distance closed set is defined with respect to the distance property between the distance closed set and the vertices of the graph. The distance closed set of a graph G is defined as follows:

Let S be a vertex subset of G. Then S is said to be *distance closed set* of G if for each vertex $u \in S$ and for each $w \in V-S$, there exists at least one vertex $v \in S$ such that $d_{<S>}(u, v) = d_G(u, w)$.The properties of the above set and related structure in graphs are in [9].

Theorem 2.1 [**9**]**:** A vertex subset D of G is said to be a distance closed set if and only if

(i) $e(u/<D>) \geq e(u/G)$ for $u \in D$;

(ii) Every non eccentric point of <D> is a cut vertex.

Theorem 2.2 [**9**]**:** A graph G, which is not an odd path, is distance closed if and only if

(i) G is a unique eccentric point graph and;

(ii) Every vertex with eccentricity at most d-1 is a cut vertex, where d is the diameter of G.

Theorem 2.3 [**9**]**:** A graph G is 0-distance closed graph if and only if G is one of the following

(i) G is P_{2n+1};

(ii) G is a ciliate.

3. Main Results

In this paper, we define a new domination parameter namely, distance closed domination as follows.

3.1 Distance Closed Dominating Sets in Graphs

Definition 3.1

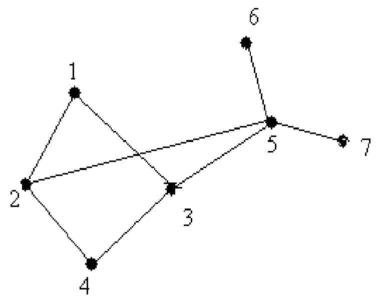
A subset $S \subseteq V(G)$ is said to be a distance closed dominating (D.C.D) set, if

(i) <S> is distance closed;

(ii) S is a dominating set.

The cardinality of a minimum D.C.D set of G is called the distance closed domination number of G and is denoted by γ_{dcl}.

Clearly, from the definition, $1 \leq \gamma_{dcl} \leq p$ and graph with $\gamma_{dcl}=p$ is called a 0-distance closed dominating graph. Also, if S is a D.C.D set of G then the complement V-S need not be a D.C.D set of G. For example



Here, S = { 1,3,5,6} form a D.C.D set but V-S ={2,4,7} is not a D.C.D set.

The following theorems give the bounds of some special class of graphs:

Theorem 3.1: If T is a tree with number of vertices $p \geq 2$, then $\gamma_{dcl}(T) = p-k+2$, where k is the number of pendant vertices in T.

Proof: The D.C.D set, S of a tree T must contain the diametrical eccentric points of T and also the induced sub graph < S > is a tree and it contains exactly the two pendant vertices of T. Hence $\gamma_{dcl}(T) = p-k+2$.

Theorem 3.2: Let G be a self centered graph of diameter 2.Then $\gamma_{dcl}(G) \leq \delta+2$.

Proof: Any δ degree vertex, vertices in first neighborhood of that vertex and a vertex in the second neighborhood form a D.C.D set of G. Hence $\gamma_{dcl}(G) \leq \delta+2$.

Theorem 3.3: Let G be a graph and S be a D.C.D set of G. Then the radius of the induced graph <S> is at least r, where r is the radius of G.

Proof: Let u be a point in the D.C.D set S with e(u)=r-1 in <S>. Then $e(u) \leq r-1$ in G(from theorem 2.1).This implies that radius of G is at most r-1, a contradiction to radius of G is r. Hence, there is no point in the D.C.D set, which has eccentricity r-1 in the induced graph and hence the radius of <S> is at least r.

Corollary 3.1: Let G be a graph with radius r and diameter d. Let S be a D.C.D set of G. Then the diameter of the induced graph <S>, induced by S is at least r.

Proof: Proof follows from Theorem 2.1 and Theorem 3.3.

Theorem 3.4: Let G be a graph of order p. Then $\gamma_{dcl}(G)=2$ if and only if G has at least two vertices of degree p-1.

Proof: Assume that $\gamma_{dcl}(G)=2$. Let D= {u,v} be the distance closed dominating set of G. Then, we claim that d(u)=d(v)=p-1. As e(u)=e(v)=1 in <D>, e(u)≤1 and e(v)≤1 in G(from theorem 2.1).This implies that, e(u)=e(v)=1 in G. That is, d(u)=d(v)=p-1.

Conversely, let d(u)=d(v)=p-1in G. That is e(u)=e(v)=1.Then clearly {u,v} will form the distance closed dominating set of G. Hence $\gamma_{dcl}(G)=2$.

Proposition 3.1: If G is a (p,q) graph and $\gamma_{dcl}(G)=2$, then q≥ (4p-6)|2.

Proof: Let G be a (p,q) graph with $\gamma_{dcl}(G)=2$. Then any two vertices of degree p-1 of G belong to the distance closed dominating set D of G and d(u) ≥2 for all u∈V-D.

$$\begin{aligned} \text{Therefore, } 2q &= \sum_{u\in V(G)} d(u) \\ &= \sum_{u\in D} d(u) + \sum_{u\in V-D} d(u) \\ &\geq 2(p-1) + (p-2)(2) \\ &= 2p-2+2p-4 \\ &= 4p-6 \end{aligned}$$

Hence, q ≥ (4p-6)|2.

Theorem 3.5: Let G be a graph of order p. If G has only one vertex of degree p-1, then $\gamma_{dcl}(G)=3$.

Proof: Let v be a vertex of degree p-1 in G. Then clearly, r(G)=1 and the diameter of G is 2. Hence, any two non adjacent vertices of eccentricity 2 together with v will form a distance closed dominating set of G and hence $\gamma_{dcl}(G)=3$.

Corollary 3.2: If G is a graph with $\gamma_{dcl}(G)=3$, then the diameter of G is 2.

Proof: Proof follows from Theorem 3.5.

Proposition 3.2: If G is a (p,q) graph and $\gamma_{dcl}(G)=3$, then q≥(p-1).

Proof: Let G be a (p,q) graph with $\gamma_{dcl}(G)=3$. Then G has exactly a vertex of degree p-1 and that vertex belongs to the distance closed dominating set D of G and d(u) ≥1 for all u∈V-D.

$$\begin{aligned} \text{Therefore, } 2q &= \sum_{u\in V(G)} d(u) \\ &= \sum_{u\in D} d(u) + \sum_{u\in V-D} d(u) \\ &\geq 1(p-1)+ (p-1)(1) \\ &= p-1+p-1 \\ &= 2p-2 \end{aligned}$$

Hence, q ≥(p-1).

Theorem 3.6: If a graph G is connected and diam(G) ≥3, then $\gamma_{dcl}(\overline{G})$=4.

Proof: Since diam(G) ≥3, $\overline{G}$ has a dominating edge. Hence, $\gamma_{dcl}(\overline{G})$=4.

Nordhaus-Gaddum results for distance closed domination number:

Theorem 3.7: For any connected graph G such that $\overline{G}$ is also connected, $\gamma_{dcl}(G)+\gamma_{dcl}(\overline{G})\leq p+4$, where $\gamma_{dcl}(G)$ and $\gamma_{dcl}(\overline{G})$are the cardinality of minimal distance closed dominating set of G and $\overline{G}$ respectively.

Proof: Let G be a connected graph G such that $\overline{G}$ is also connected.

Case 1: For r=1, d=1 or r=1, d=2.

Clearly, there is no graph G with the above cases such that both G and $\overline{G}$ are connected.

Case 2: For r=2 and d=2.

Consider a vertex v. Clearly $\{v\}\cup N_1(v)\cup$ a vertex in $N_2(v)$ forms a distance closed dominating set of G. In order to have both G and $\overline{G}$ are connected, there exists at least one vertex u in $N_1(u)$ having eccentric point in $N_2(u)$.In this case we have two sub cases.

Sub case(i): G has a dominating edge.

If G is a self centered graph of diameter 2 having a dominating edge, then clearly $\gamma_{dcl}(G)$=4 and $\overline{G}$ is of diameter≥3. Hence, $\gamma_{dcl}(\overline{G})\leq$ p and hence $\gamma_{dcl}(G)+\gamma_{dcl}(\overline{G})\leq p+4$.

Sub case(ii)): G has no dominating edge.

If G is a self centered graph of diameter 2 without dominating edge then $\overline{G}$ is also self centered graph of diameter 2.Then, $\gamma_{dcl}(G)\leq\delta(G)+2$ and $\gamma_{dcl}(\overline{G})\leq\delta(\overline{G})+2$(By theorem 3.2)

Therefore, $\gamma_{dcl}(G)+\gamma_{dcl}(\overline{G})\leq\delta(G)+2+\delta(\overline{G})+2=\delta(G)+\Delta(G)+4=p-1+4$

Hence $\gamma_{dcl}(G)+\gamma_{dcl}(\overline{G})\leq p+3$.

Case 3: For r≥2 and d≥3.

In G, there must exist at least two vertices, which has distance greater than or equal to 3. Then, clearly that two vertices form a dominating set for $\overline{G}$ and the eccentricity of that two vertices are 2. Therefore, $\gamma_{dcl}(\overline{G})$=4 whenever diam(G) ≥3.

Hence $\gamma_{dcl}(G)+\gamma_{dcl}(\overline{G})\leq p+4$.

Theorem 3.8: For any connected graph G such that $\overline{G}$ is also connected, $\gamma_{dcl}(G)+\gamma_{dcl}(\overline{G})=p+4$ if and only if one of the graphs G or $\overline{G}$ is a 0- distance closed dominating graph.

Proof: Let G be any graph. Assume that $\gamma_{dcl}(G)+\gamma_{dcl}(\overline{G})=p+4$. Then clearly, $\gamma_{dcl}(G)$ and $\gamma_{dcl}(\overline{G})$ are not less than or equal to 4. Thus either d(G) or $d(\overline{G})$ is greater than or equal to 3. Without loss of generality assume that d(G) is greater than or equal to 3.Then clearly, $\gamma_{dcl}(\overline{G})=4$ and hence $\gamma_{dcl}(G)+\gamma_{dcl}(\overline{G})=p+4$ implies that $\gamma_{dcl}(G)=p$, that is G is a 0-distance closed dominating graph.

Conversely, assume that G or $\overline{G}$ is a 0- distance closed dominating graph. Without loss of generality, let G be a 0- distance closed dominating graph. Then $\gamma_{dcl}(G)=p$ and $d(G)\geq 3$.This implies that $\gamma_{dcl}(\overline{G})=4$. Hence $\gamma_{dcl}(G)+\gamma_{dcl}(\overline{G})=p+4$.

Remark 3.1: The above bound is attainable for ciliates and paths.

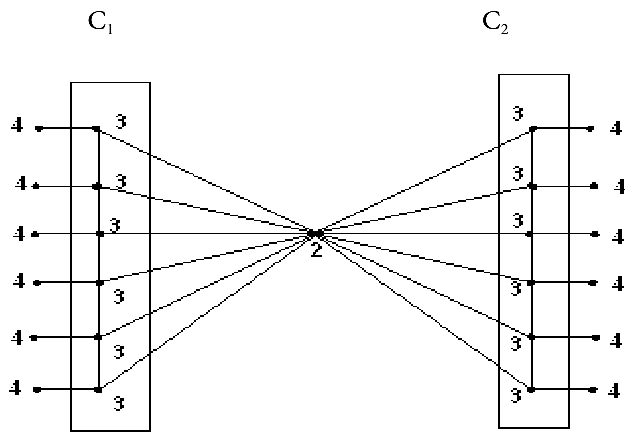
Theorem 3.9: There is no graph G such that both G and $\overline{G}$ are 0- distance closed dominating graphs.

Proof: Since all the 0-D.C.D graph are with diameter ≥ 3 and there is no graph for which both G and $\overline{G}$ are with diameter ≥ 3, we have the result.

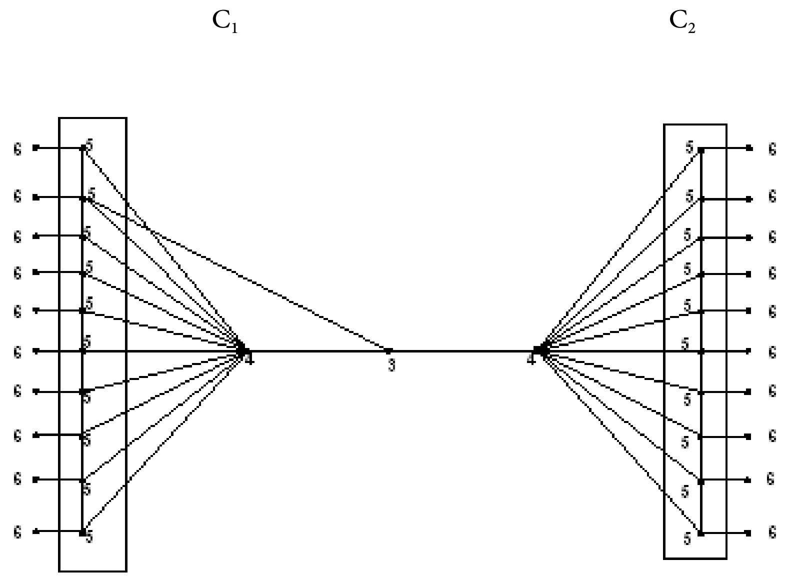
Remark 3.2:

The D.C.D set of a graph G need not be connected. We can see this in graphs with higher diameter. In particular, for any graph G with diameter≤3, the minimum D.C.D set must be connected. But graphs with diameter≥4 can have a disconnected minimum D.C.D set. For example, the following structure of graphs having a disconnected minimum D.C.D set.

Graphs having the following structure must have a disconnected minimum D.C.D set:



Graph with diameter 4 having a disconnected minimum D.C.D set $\{C_1, C_2\}$



Graph with diameter 6 having a disconnected minimum D.C.D set $\{C_1, C_2\}$

Similarly, we can construct graphs with diameter ≥7. Also these structure of graphs are having $\gamma=\gamma_{dcl}$.

Open problem:

1. For any graph G with diameter ≤3, the minimum D.C.D set must be connected- prove.
2. For every d ≥4, there exists at least one graph with diameter d, which has a disconnected minimum D.C.D set - prove.

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