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Odd and Even Weak Convex Critical Graph and Domatic partition of Graphs

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Abstract: *In a graph $G = (V, E)$, a set $D \subset V$ is a weak convex dominating(WCD) set if each vertex of V-D is adjacent to at least one vertex in D and $d_{<D>}(u,v) = d_G(u,v)$ for any two vertices u, v in D. A weak convex domination set D is said to be odd(even) W.C.D set, if for any vertex $u \in V$-D, there exists $v \in D$ at odd(even) distance from u. The domination number $\gamma_{oc}(G)$ is the smallest order of a odd weak convex dominating set of G and the domination number $\gamma_{ec}(G)$ is the smallest order of a odd weak convex dominating set of G. In this paper we study the change in the behaviour of even weak convex domination number with respect to addition of edges in the respective graph and also the domatic partition of a graph with respect to even dominating sets of a graph..*

Keywords: *domination number, distance, eccentricity, radius, diameter, self-centered, neighbourhood, weak convex dominating set, odd weak convex dominating set, even weak convex dominating set.*

1. Introduction

Graphs discussed in this paper are undirected and simple. Unless otherwise stated the graphs which we consider are connected graphs only. For a graph G, let V(G) and E(G) denote its vertex and edge set respectively. The *degree* of a vertex v in a graph G is denoted by $\deg_G(v)$. The length of any shortest path between any two vertices u and v of a connected graph G is called the *distance between u and v* and is denoted by $d_G(u, v)$. The distance between two vertices in different components of a disconnected graph is defined to be ∞. For a connected graph G, the *eccentricity* $e_G(v) = \max\{d_G(u, v): u \in V(G)\}$. If there is no confusion, we simply use the notion $\deg(v)$, $d(u, v)$ and $e(v)$ to denote degree, distance and eccentricity respectively for the concerned graph. The minimum and maximum eccentricities are the *radius* and *diameter* of G, denoted r(G) and diam(G) respectively. When these two are equal, the graph is called *self-centered* graph with radius r, equivalently is *r self-centered*. A vertex u is said to be an *eccentric vertex* of v in a graph G, if $d(u, v) = e(v)$. In general, u is called an eccentric vertex, if it is an eccentric vertex of some vertex. For $v \in V(G)$, the *neighbourhood* $N_G(v)$ of v is the set of all vertices adjacent to v in G. The set $N_G[v] = N_G(v) \cup \{v\}$ is called the *closed neighbourhood* of v. We define

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the set $N_G[S] = \bigcup_{u \in S} N_G(u)$ -S as the open *neighbourhood* of a set $S \subseteq V(G)$. A set S of edges in a graph is said to be *independent* if no two of the edges in S are adjacent. An edge e = (*u*, *v*) is a *dominating edge* in a graph G if every vertex of G is adjacent to at least one of *u* and *v*.

The concept of domination in graphs was introduced by Ore. A set $D \subseteq V(G)$ is called *dominating set* of G if every vertex in V(G)-D is adjacent to some vertex in D. D is said to be a *minimal* dominating set if D-{*v*} is not a dominating set for any $v \in D$. The *domination number* γ(G) of G is the minimum cardinality of a dominating set. We call a set of vertices a γ-set if it is a dominating set with cardinality γ(G). Different types of dominating sets have been studied by imposing conditions on the dominating sets. A dominating set D is called *connected* (*independent*) dominating set if the induced subgraph <D> is connected (independent). D is called a *total dominating set* if every vertex in V(G) is adjacent to some vertex in D.

A cycle of D of a graph G is called a *dominating cycle* of G, if every vertex in V – D is adjacent to some vertex in D. A dominating set D of a graph G is called a *clique dominating set* of G if <D> is complete. A set D is called an *efficient dominating set* of G if every vertex in V – D is adjacent to exactly one vertex in D. A set $D \subseteq V$ is called a *global dominating set* if D is a dominating set in G and $\overline{G}$. A set D is called a *restrained dominating set* if every vertex in V(G)-D is adjacent to a vertex in D and another vertex in V(G)-D. A set D is a *weak convex dominating set* if each vertex of V-D is adjacent to at least one vertex in D and the distance between any two vertices *u* and *v* in the induced graph <D> is equal to that of those vertices *u* and *v* in G. By $\gamma_c, \gamma_i, \gamma_t, \gamma_o, \gamma_k, \gamma_e, \gamma_g, \gamma_r$ and γ_{wc}, we mean the minimum cardinality of a connected dominating set, independent dominating set, total dominating set, cycle dominating set, clique dominating set, efficient dominating set, global dominating set and restrained dominating set respectively.

In this paper we introduce a new dominating set called acyclic weak convex dominating set of a graph through which we studied the properties of the graph such as variation in radius and diameter of the graph. We find upper and lower bounds for the new domination number in terms of various already known parameters. Also we studied several interesting properties like Nordhaus-Gaddum type results relating the graph and its complement.

2. Odd and Even Weak Convex Critical Graph

Definition 2.1:

A graph is said to be k - E.W.C.D critical graph if $\gamma_{ec}(G+e) < \gamma_{ec}(G)$ and $\gamma_{ec}(G) = k$, for any edge $e \notin E(G)$.

A graph is said to be k - O.W.C.D critical graph if $\gamma_{oc}(G+e) < \gamma_{oc}(G)$ and $\gamma_{oc}(G) = k$, for any edge $e \notin E(G)$.

Proposition 2.1:

There cannot be any 2–E.W.C.D critical graph.

Proof:

Since the only graph for which $\gamma_{ec} = 1$ is K_1 and 2- E.W.C.D critical means if we join any two non-adjacent vertices will yield $\gamma_{ec} = 1$. Thus there cannot be any 2-E.W.C.D critical graph.

Proposition 2.2:

A graph G is 3-E.W.C.D critical $\Leftrightarrow$ any one of the following holds good for any non-adjacent vertices u and v

(i) $N(u) \cup N(v) = V(G) - \{u,v\}$ and $N[u] \cap N[v] = \phi$

(ii) There exists a vertex w such that $N(u) \cup N(w) = V(G) - \{v\}$ and $N(u) \cap N(w) = \phi$ (or) $N(v) \cup N(w) = V(G) - \{u\}$ and $N(v) \cap N(w) = \phi$.

Proposition 2.3:

Any two pendant vertices cannot have a same support in a k-E.W.C.D critical graph.

Proof:

Since, if we join those pendant vertices, it will not reduce the domination number of the graph.

Proposition 2.4:

Any 3-E.W.C.D critical graph has at the most 2 pendant vertices.

Proof:

Let G be a 3- E.W.C.D critical graph. Let x, y and z be the pendant vertices of G and u, v and w be the supports of them respectively. Then clearly {u, v, w} is the only dominating set of G. If we join x and y then still G + xy have a 3-dominating set which is either {u, v, w} or {x, u, w} or {y, v, w}. Thus $\gamma_{ec}(G + xy) = 3$ only, which is a contradiction to G is 3-E.W.C.D critical. Hence G has at most 2 pendant vertices only.

Proposition 2.5:

The diameter of 3-E.W.C.D critical graph is at most 3.

Proof:

Let G be a 3-E.W.C.D critical graph.

Let u and v be any two non-adjacent vertices of G. Then the graph G+uv has a two dominating set, which contains any one of the vertex u or v. Without loss of generality assume that {u, w} dominates G+uv. If u dominates $N_1(v)$, then d(u, v) = 2. If w dominates $N_1(v)$, then d(u, v)$\leq$3. Hence the proof.

Proposition 2.6:

Any 3-E.W.C.D critical graph is a block.

Proof:

Let G be a 3-E.W.C.D critical graph. Let u be a cut vertex of G.

Let $C_1, C_2, \ldots\ldots, C_n$ be the components of G – u. Clearly all the vertices of C_i's are not adjacent to u. Therefore, there exists a vertex v in some component say C_1, which is of distance greater than or equal to 2. Also we have no vertex of distance greater than 3 from u, otherwise distance between that vertex and any other vertex will become greater than 4, which is a contradiction to the previous proposition. Thus we have d(u,v) = 2 in G. Also we have no other vertex in $C_2, \ldots\ldots. C_n$ is of distance greater than or equal to 2 from u, otherwise distance between v and those vertices will become more than 4. Hence all the vertices of $C_2, \ldots, C_n$ are adjacent to u. Now each of the components forms a clique, otherwise if we join any two non-adjacent vertices of any of the components $C_2, \ldots, C_n$ they will not reduce the domination number, which is a contradiction to G is critical. Therefore, each of the components $C_2, \ldots, C_n$ forms a clique. Now if we join any two vertices each from one of the components $C_2, \ldots, C_n$ will not affect the domination. Hence $C_2, \ldots, C_n$ form a single component only. Therefore, we have only two components C_1 and C_2 in which $C_2 \cup \{u\}$ form a clique.

Now join any two vertices $v \in C_1 \cap N_1(u)$ and $w \in C_2$. Either v or w must be in the 2-dominating set of G+uw. Clearly {vx/x$\neq$w} cannot be a dominating set for G+vw. If x $\neq$ u then vx cannot dominate C_2. Also, if uv dominate G+vw it can dominate G also. Clearly uw also cannot form an even weak convex dominating set for G+vw, since there exists a vertex in C_1, which is at distance 2 from u (by previous arguments) as well as from w also. Clearly, {v, w} also cannot form an even dominating set as they are both adjacent to u. Hence there exists no 2-dominating set exist for G+vw. Also any singleton vertex cannot form an even dominating set for G+vw. Hence there cannot be two component C_1 and C_2 in G. Hence u will not be a cut vertex. Thus, G is a block.

Proposition 2.7:

A graph G is k-weak convex domination critical graph if and only if G is k-odd weak convex domination critical graph.

3. Even Weak Convex Domatic Number

Definition:

An Even Weak Convex Domatic Number d_{ec} of a graph G is the maximum partition $\{V_1, V_2, \ldots, V_n\}$ of V(G) such that each V_i, $1 \leq i \leq n$ is an E.W.C.D. set of G.

Observations:

3.1 : $d_{ec}(K_p) = 1$.

3.2 : If $d_{ec}(G) \geq 2$, then G is a block

3.3 : For any tree T, $d_{ec}(T) = 1$.

3.4 : $d_{ec}(K_{m,n}) = \begin{cases} 1 & \text{if } m \text{ or } n = 0 \\ 2 & \text{otherwise} \end{cases}$

3.5 : $d_{ec}(C_n) = \begin{cases} 1 & \text{if } p \neq 4 \\ 2 & \text{if } p = 4 \end{cases}$

3.6 : For any 0-W.C.D graph G, $d_{ec}(G) = 1$.

3.7 : For any geodetic block of diameter 2, $d_{ec}(G) \leq 2$.

3.8 : For Petersen graph, $d_{ec} = 2$, for other geodetic blocks $d_{ec} = 1$.

Proposition 3.1:

For any graph G, $d_{ec}(G) \leq \delta + 1$.

Proposition 3.2:

$d_{ec}(G) = \delta + 1 \Leftrightarrow G = K_1$

Proof:

Let G be a graph on p vertices with $d_{ec}(G) = \delta + 1$.

Let v be a vertex with degree δ. Then the E.W.C.D set in the domatic partition containing v does not contain any vertex other than v. (otherwise degree of v must be increased to $\delta+1$). This implies that v itself form a dominating set for G. Since a E.W.C.D set must contain at least two vertices, $G = K_1$.

Proposition 3.3:

For any graph G with diameter $d \geq 3$, $d_{ec} \leq [n/(d-1)]$

Proposition 3.4:

For any graph G with radius r, $d_{ec} \leq [n/(2r-2)]$

Proposition 3.5:

For any graph $G \neq K_1$, $d_{ec} \leq \delta$.

Proposition 3.6:

For any graph G, $d_{ec}(G) + d_{ec}(\overline{G}) \leq p + 1$.

Proposition 3.7:

$d_{ec}(G) + d_{ec}(\overline{G}) = p + 1 \Leftrightarrow G = K_1$.

Proposition 3.8:

For any graph $G \neq K_1$, $d_{ec}(G) + d_{ec}(\overline{G}) \leq p - 1$.

Proposition 3.9:

For any cycle C_n, $n \geq 5$, $d_{ec}(C_n) + d_{ec}(\overline{C_n}) = 2$

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