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# On The Complement of the Boolean Function Graph $B(\overline{K_p}$, NINC, L(G)) of a Graph

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***Abstract:*** *For any graph G, let V(G) and E(G) denote the vertex set and edge set of G respectively. The Boolean function graph $B(\overline{K}_p$, NINC, L(G)) of G is a graph with vertex set V(G)∪E(G) and two vertices in $B(\overline{K}_p$, NINC, L(G)) are adjacent if and only if they correspond to two adjacent edges of G or to a vertex and an edge not incident to it in G. For brevity, this graph is denoted by $B_2(G)$. In this paper, structural properties of the complement $\overline{B_2}(G)$ of $B_2(G)$ including traversability and eccentricity properties are studied. Also covering, independence and chromatic numbers are determined.*

## 1. Introduction

Graphs discussed in this paper are undirected and simple graphs. For a graph G, let V(G) and E(G) denote its vertex set and edge set respectively. *Eccentricity* of a vertex $u \in V(G)$ is defined as $e_G(u) = \max \{d_G(u, v): v \in V(G)\}$, where $d_G(u, v)$ is the distance between u and v in G. If there is no confusion, then we simply denote the eccentricity of vertex v in G as e(v) and d(u, v) to denote the distance between two vertices u, v in G respectively. The minimum and maximum eccentricities are the *radius* and *diameter* of G, denoted r(G) and diam(G) respectively. When diam(G) = r(G), G is called a *self-centered* graph with radius r, equivalently G is r-self-centered. A vertex u is said to be an eccentric point of v in a graph G, if d(u, v) = e(v). In general, u is called an *eccentric point*, if it is an eccentric point of some vertex. We also denote the $i^{th}$ neighborhood of v as $N_i(v)=\{u \in V(G) : d_G(u , v) = i\}$. A connected graph G is said to be *geodetic*, if a unique shortest path joins any two of its vertices.

A vertex and an edge are said to *cover* each other, if they are incident. A set of vertices, which covers all the edges of a graph G is called a *point cover* for G. The smallest number of vertices in any point cover for G is called its *point covering number* and is

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denoted by $\alpha_0(G)$ or $\alpha_0$. A set of vertices in G is *independent*, if no two of them are adjacent. The largest number of vertices in such a set is called the *point independence number* of G and is denoted by $\beta_0(G)$ or $\beta_0$.

The *Boolean function graph* $B(\overline{K_p}, NINC, L(G))$ G is a graph with vertex set $V(G)\cup E(G)$ and two vertices in $B(\overline{K_p}, NINC, L(G))$ are adjacent if and only if they correspond to two adjacent edges of G or to a vertex and an edge not incident to it in G. For brevity, this graph is denoted by $B_2(G)$. In other words, $V(B_2(G)) = V(G)\cup V(L(G))$; and $E(B_2(G)) = [E(T(\overline{G}))-(E(\overline{G})\cup E(\overline{L(G)}))]\cup E(L(G)$, where $\overline{G}$, L(G) and T(G) denote the complement, the line graph and the total graph of G respectively. The vertices of G and L(G) in $B_2(G)$ are referred as point and line vertices respectively and the line vertex in $B_2(G)$ corresponding to an edge e in G is denoted by $e'$. In this paper, we study structural properties of the complement $\overline{B_2(G)}$ of $B_2(G)$ including traversability and eccentricity properties. The definitions and details not furnished in this paper may found in [1].

The mixed relations of incident, non-incident, adjacent and non-adjacent can be used to analyze nature of clustering of elements of communication networks. The concept of chromatic number could be used in a particular type of clustering network such that each cluster is either independent in that nework. Also any other clustering of network with each cluster having at lest one colour class element of vertices network.

## 2. Prior Results

**Theorem 2.1 [1]:** For any nontrivial connected graph G, $\alpha_0 + \beta_0 = p$, where p is the number of vertices in G.

## 3. Main Results

The following elementary properties of the complement $\overline{B_2(G)}$ of the Boolean function graph $B_2(G)$ of a graph G are immediate. Let G be a (p, q) graph.

**Observation:**

**3.1:** Degree of a point vertex v in $\overline{B_2(G)}$ is $p - 1 + \deg_G(v)$ and the degree of line vertex $e'$ is $q + 1 - \deg_{L(G)}(e')$.

**3.2:** $\overline{B_2(G)}$ is connected, for any graph G.

**3.3:** $\overline{B_2(G)}$ is biregular if and only if G is regular and is regular if and only if G $\cong nK_1$, for $n \geq 2$.

**3.4:** No vertex of $\overline{B_2(G)}$ is a cut-vertex.

**3.5:** Girth of $\overline{B_2(G)}$ is 3.

**3.6:** Maximum number of edge disjoint triangles in $\overline{B_2(G)}$ is q and each vertex of $\overline{B_2(G)}$ lies on a triangle.

**3.7:** Each edge of $\overline{B_2(G)}$ lies on a triangle if and only if each edge of $\overline{L(G)}$ lies on a triangle. In this case, $L(\overline{B_2(G)})$ is Hamiltonian.

**3.8:** If $K_2 \cup K_1$ is a sub graph of G, then $\overline{B_2(G)}$ contains $K_4$-e as an induced sub graph and hence not geodetic. Therefore, $\overline{B_2(G)}$ is geodetic if and only if G $\cong nK_1$ or $K_2$, for $n \geq 2$.

In the following, a characterization of $\overline{B_2(G)}$ to be Eulerian is given.

**Theorem 3.1:** Let G be any (p, q) graph with q odd. Then $\overline{B_2(G)}$ is Eulerian if and only if degree of each vertex in G is of same parity.

**Proof:** Assume q is odd and $\overline{B_2(G)}$ is Eulerian. Then degree of each vertex in $\overline{B_2(G)}$ is even.

**Case(i): p is odd.**

Since degree of a point vertex v in $\overline{B_2(G)}$ is even, $p - 1 + deg_G(v)$ is even and hence $deg_G(v)$ is even, for all $v \in V(G)$

**Case(ii): p is even.**

Then $p - 1 + deg_G(v)$ is even implies that $deg_G(v)$ is odd, for all $v \in V(G)$. Hence, each vertex in G is of same parity.

Conversely, assume q is odd and degree of each vertex in G is of same parity. Then degree of each vertex in L(G) is even and hence degree of a line vertex $e'$ in $\overline{B_2(G)}$ is $q + 1 - deg_{L(G)}(e')$ is even. Let v be a point vertex in $\overline{B_2(G)}$. If $deg_G(v)$ is odd, for all v in G, then since the number of odd degree vertices is even, p is even and hence the degree of v in $\overline{B_2(G)}$ is $p - 1 + deg_G(v)$ is even. If $deg_G(v)$ is even for all v in G, since q is odd, p is also odd and hence degree of v in $\overline{B_2(G)}$ is even Thus, degree of a point vertex is even. Hence, $\overline{B_2(G)}$ is Eulerian.

**Theorem 3.2:** If $\overline{L(G)}$ is Hamiltonian, then $\overline{B_2(G)}$ is Hamiltonian.

**Proof:** Assume $\overline{L(G)}$ is Hamiltonian. Then there exists a Hamiltonian cycle, say $e_1'$ $e_2'\ldots e_q'$ $e_1'$ in $\overline{L(G)}$. Let $v_1$ and $v_q$ be any two vertices in G, incident with the edges in G corresponding to the vertices $e_1'$ and $e_q'$ in the Hamiltonian cycle respectively. In the above Hamiltonian cycle, place $v_q$, $v_1$ in between $e_q'$ and $e_1'$ and then place the remaining point vertices in between $v_q$ and $v_1$. This is possible, since the sub graph of $\overline{B_2}(G)$ induced by all point vertices is complete. This will form a Hamiltonian cycle of $\overline{B_2}(G)$ and hence $\overline{B_2}(G)$ is Hamiltonian.

**Theorem 3.3:** Let G be any (p, q) graph with p > q. If $\Delta_e(G) \leq \delta(G) + 1$, then $\overline{B_2}(G)$ is Hamiltonian, where $\Delta_e(G)$ is the maximum degree of L(G).

**Proof:** This theorem is proved by finding the closure of $\overline{B_2}(G)$. Since the sub graph of $\overline{B_2}(G)$ induced by the point vertices is complete, any two point vertices in $\overline{B_2}(G)$ are adjacent. Since $\Delta_e(G) \leq \delta(G) + 1$, the sum of the degrees of any two nonadjacent point, line vertices in $\overline{B_2}(G)$ exceeds $p + q - 1$ and they can be made adjacent. Therefore, in the closure of $\overline{B_2}(G)$, any two point vertices are adjacent and any two point, line vertices are adjacent. Construct a path in the closure of $\overline{B_2}(G)$ on 2q vertices with the initial vertex, a point vertex and the terminal vertex, a line vertex and point, line vertices occurring alternately. Then place the remaining p - q point vertices in the above path since p > q. Hence, there exists a Hamiltonian cycle in the closure of $\overline{B_2}(G)$ and is Hamiltonian. Thus, $\overline{B_2}(G)$ is Hamiltonian.

In the following, the radius and diameter of $\overline{B_2}(G)$ are determined. For simplicity, $\overline{d_2}(u)$, $\overline{e_2}(v)$ and $\overline{d_2}(u, v)$ are used to denote the degree of a vertex u, the eccentricity of a vertex v and the distance between the vertices u and v in $\overline{B_2}(G)$ respectively.

**Theorem 3.4:** Let G be any graph not totally disconnected with at least three vertices. Then diam($\overline{B_2}(G)$) = 2.

**Proof:** Since any two point vertices in $\overline{B_2}(G)$ are adjacent, distance between any two point vertices is 1. Let $e_1'$ and $e_2'$ be any two line vertices in $\overline{B_2}(G)$ and $e_1$ and $e_2$ be the corresponding edges in G. Then

$$\overline{d_2}(e_1', e_2') = \begin{cases} 1, & \text{if } (e_1, e_2) \notin E(G); \\ 2, & \text{if } (e_1, e_2) \in E(G). \end{cases}$$

Let v, $e'$ be a point, line vertex in $\overline{B_2}(G)$ respectively and e be the edge in G corresponding to $e'$. Then

$$\overline{d_2}(v, e') = \begin{cases} 1, & \text{if } v \in e; \\ 2, & \text{if } v \notin e. \end{cases}$$

Since G has at least 3 vertices and not totally disconnected, it follows that diameter of $\overline{B_2}(G)$ is 2.

**Corollary 3.4.1:** $\overline{B_2}(G)$ is bi-eccentric with radius 1 if and only if $G \cong K_{1,n} \cup mK_1$, $K_2 \cup tK_1$, for $n \geq 2$, $m \geq 0$ and $t \geq 1$.

**Proof:** Radius of $\overline{B_2}(G)$ is 1 if and only if there exists a vertex v in G such that each edge in G is incident with v. That is, $G \cong K_{1,n} \cup mK_1$, $K_2 \cup tK_1$, for $n \geq 2$, $m \geq 0$ and $t \geq 1$.

**Corollary 3.4.2:** Let G be a graph with at least three vertices and not totally disconnected. $\overline{B_2}(G)$ is complete if and only if $G \cong nK_1$ or $K_2$, for $n \geq 2$.

**Corollary 3.4.3:** If G is none of the graphs $K_{1,n} \cup mK_1$, $K_2 \cup tK_1$, $nK_1$ and $K_2$, for $n \geq 2$, $m \geq 0$ and $t \geq 1$, then $\overline{B_2}(G)$ is self-centered with radius 2.

In the following, point independence number, point covering number and chromatic number for $\overline{B_2}(G)$ are obtained.

**Theorem 3.5:** For any connected graph G, $\beta_0(\overline{B_2}(G)) = \Delta(G)$ or $\Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G.

**Proof:** Since $\overline{L}(G)$ is an induced sub graph of $\overline{B_2}(G)$ and any two point vertices in $\overline{B_2}(G)$ are adjacent, a maximum independent set in $\overline{L}(G)$ together with a point vertex is a maximum independent set in $\overline{B_2}(G)$. If $G \cong C_3$ or G contains at least $\Delta(G) + 2$ vertices, then $\beta_0(\overline{B_2}(G))) = \Delta(G) + 1$. Otherwise, $\beta_0(\overline{B_2}(G)) = \Delta(G)$.

Next, point independence number for $\overline{B_2}(G)$ is obtained, when G is a disconnected graph.

**Theorem 3.6:** Let G be any disconnected graph (not totally disconnected) with $\Delta(G) = 2$. If one of the components of G is $C_3$, then $\beta_0(\overline{B_2}(G)) = 4$.

**Proof:** Let G contain $C_3$ as one of its components. Then the set of line vertices corresponding to the edges in $C_3$ and a point vertex corresponding to a vertex in any other component is the maximum independent set in $\overline{B_2}(G)$. Hence, $\beta_0(\overline{B_2}(G)) = 4$.

**Remark 3.1:**

(i). If $G \cong C_3 \cup K_2$, then $\beta_0(\overline{B_2}(G)) = 4$.

(ii). If G is disconnected and if either $\Delta(G) = 2$ and none of the components is $C_3$ or if $\Delta(G) > 3$ or $\Delta(G) = 1$, then $\beta_0(\overline{B_2}(G)) = \Delta(G)$ or $\Delta(G) + 1$.

**Theorem 3.7:** For any connected graph G, $\alpha_0(\overline{B_2}(G)) = p + q - \Delta(G)$ or $p + q - \Delta(G) - 1$.

**Proof:** This follows from $\alpha_0(\overline{B_2}(G)) + \beta_0(\overline{B_2}(G)) = p + q$ and Theorem 3.5.

Similarly, $\alpha_0(\overline{B_2}(G))$ can be obtained by using Theorem 3.6 and Theorem 2.1.

**Theorem 3.8:** Let G be any disconnected graph (not totally disconnected) with $\Delta(G) = 2$. If one of the components of G is $C_3$, then $\alpha_0(\overline{B_2}(G)) = p + q - 4$.

Next, the chromatic number $\chi$ of $\overline{B_2}(G)$ is determined.

**Theorem 3.9:** For any (p, q) graph with $p \geq 3$, $\chi(\overline{B_2}(G)) = p$.

**Proof:** The sub graph of $\overline{B_2}(G)$ induced by all the p point vertices is complete. Hence $\chi(\overline{B_2}(G)) \geq p$. It is to be noted that $V(\overline{L}(G))$ can be partitioned into at most p - 1 independent sets. Since $\overline{L}(G)$ is an induced sub graph of $\overline{B_2}(G)$ and any line vertex in $\overline{B_2}(G)$ is adjacent to exactly two point vertices, any p-coloring can be extended to $\overline{B_2}(G)$. Thus, $\chi(\overline{B_2}(G)) = p$.

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