International Journal of Engineering Science, Advanced Computing and Bio-Technology
Vol. 1, No. 2, April –June 2010, pp. 80-95

Dom-Chromatic Sets in Bipartite Graphs

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Abstract: *Let G be a simple graph with vertex set V and edge set E. A subset S of V is said to be a dom-chromatic set (or dc-set) if S is a dominating set and the chromatic number of the graph induced by S is the chromatic number of G. The minimum cardinality of a dom-chromatic set in a graph G is called the dom-chromatic number (or dc- number) and is denoted by $\gamma_{ch}(G)$. In this paper, bounds for dom-chromatic numbers for bipartite graphs are discussed.*

Keywords: *Dominating set, domination number, dom-chromatic set (or dc-set), dom-chromatic number (or dc-number).*

1. Introduction

In this paper, we discuss finite, simple undirected graphs. For any graph G, V denotes the vertex set, E denotes the edge set and p denotes the number of vertices. For any graph theoretic terminology which is not defined refer to Harary [2].

The chromatic number $\chi(G)$ is the minimum k such that vertices of G is properly k-colorable. A graph G is said to be a vertex-color-critical graph if $\chi(G - u) < \chi(G)$ for every $u \in V$ and edge-critical if $\chi(G - e) < \chi(G)$ for every $e \in E$. Clearly, edge-critical graphs are vertex-color-critical. In general, any element t of the set $V(G) \cup E(G)$ is critical if $\chi(G - t) < \chi(G)$. A graph is called a color-critical graph if each of its vertices and edges are critical. It is to be noted that the only k-critical graphs for k = 1, 2 and 3 are K_1, K_2 and odd cycles, respectively.

A set $S \subseteq V$ is a dominating set of G if for each $u \in V - S$, there exists a vertex $v \in S$ such that u is adjacent to v. The minimum cardinality of a dominating set in G is called the domination number of G, denoted by $\gamma(G)$. Harary and Haynes [3] defined the conditional domination number $\gamma(G: P)$ as the smallest cardinality of a dominating set $S \subseteq V$ such that the sub graph <S> induced by S satisfies a graph property P. A dominating set $S \subseteq V$ of G is a global dominating set if S is also a dominating set in the complement $\overline{G}$ of G.

Received: 09 June, 2009; Revised: 27 October, 2009; Accepted: 14 December, 2009

In this paper, we introduce a new conditional dominating set called dom-chromatic set or simply a dc-set which combines domination and coloring property of a graph. A subset S of V is said to be a dom-chromatic set (or dc-set) if S is a dominating set and $\chi(\langle S \rangle) = \chi(G)$. The minimum cardinality of a dom-chromatic set in a graph G is called the dom-chromatic number (or dc- number) and is denoted by $\gamma_{ch}(G)$.

2. Preliminary results

This section contains some results about domination and some preliminary results on dom-chromatic number which will be used in the next section to prove the main result.

Theorem 2.1 [5, pp 41]: If a graph G has no isolated vertices, then $\gamma(G) \le \left\lfloor \frac{p}{2} \right\rfloor$.

Theorem 2.2: [5, pp 50]: For any graph G, $\left\lfloor \frac{p}{\Delta(G)+1} \right\rfloor \le \gamma(G) \le p - \Delta(G)$.

In a dom-chromatic set, the following observations are made.

Observation 2.3:

(i) Dom-chromatic set exists for all graphs.

(ii) Vertex set V is a trivial dom-chromatic set.

(iii) If S is a dom-chromatic set of G, then each vertex of V – S is not adjacent to at least one vertex of S.

(iv) A dom-chromatic set of a graph is a global dominating set.

Proof:

(i) and (ii) follow trivially.

(iii) Suppose S is a dom-chromatic such that $x \in V - S$ is adjacent to each vertex of S, then $\chi(G) \ge \chi(\langle S \rangle) + 1 = \chi(G) + 1$ which is a contradiction.

(iv) Let S be a dom-chromatic of a graph G. From (iii), in $\overline{G}$ each vertex of V – S is adjacent to at least one vertex of S. Hence, S is a dominating set of $\overline{G}$ and the result follows.

Proposition 2.4: A dom-chromatic set S is minimal if and only if for each $u \in S$, at least one of the following conditions hold.

(i) $\chi(\langle S - u \rangle) < \chi(G)$.

(ii) S – u is not a dominating set.

The dom-chromatic number for some standard graphs:

Proposition 2.5:

(i) $\gamma_{ch}(K_n) = n$

(ii) $\gamma_{ch}(nK_1) = n$;

(iii) $\gamma_{ch}(K_{m,n}) = 2$

(iv) $\gamma_{ch}(P_n) = \begin{cases} (n+3)/3, & \text{if } n \equiv 0 \pmod 3 \\ (n+2)/3, & \text{if } n \equiv 1 \pmod 3 \\ (n+4)/3, & \text{if } n \equiv 2 \pmod 3 \end{cases}$

(v) a. If n is odd, then $\gamma_{ch}(C_n) = n$.

b. If n is even, then

$$\gamma_{ch}(C_n) = \begin{cases} (n+3)/3, & \text{if } n \equiv 0 \pmod 3 \\ (n+2)/3, & \text{if } n \equiv 1 \pmod 3 \\ (n+4)/3, & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

(vi) $\gamma_{ch}(W_n) = \begin{cases} 3, & \text{if n is odd} \\ n, & \text{if n is even.} \end{cases}$

Proposition 2.6: If G is a disconnected graph with k components $G_1, G_2, \ldots, G_k$, then $\gamma_{ch}(G) = \gamma_{ch}(G_m) + \sum_{\substack{i=1 \\ i \neq m}}^{k} \gamma(G_i)$, where $\gamma_{ch}(G_m) = \min_{1 \le i \le k} \{\gamma_{ch}(G_i): \chi(G_i) = \chi(G)\}$, for $m \in \{1, 2\ldots, k\}$.

Proposition 2.7: If G is any connected graph, then $\gamma_{ch}(G) = p - q$ if and only if $G = K_1$.

Proof: Necessary condition is trivial. Suppose that $\gamma_{ch}(G) = p - q$. Since $\gamma_{ch}(G) \ge 1$, $p - q \ge 1$. Further, as G is connected, $p - q \le 1$. Thus, $p - q = 1$ and hence $G = K_1$.

Proposition 2.8: Let D be any dom-chromatic set of G. Then $|V - D| \le \sum_{u \in D} deg(u)$.

Proposition 2.9: Let D be any dom-chromatic set of G. Then $|V - D| = \sum_{u \in D} deg(u)$ if and only if $G = pK_1$, $p \ge 1$.

Proof: If $G = pK_1$, then $D = V$ and $\deg(u) = 0$ for each $u \in D$. Then the equality holds. Now suppose that $|V - D| = \sum_{u \in D} \deg(u) = k$.

<u>Claim:</u> $k = 0$.

Suppose $k \ge 1$, then two cases arise.

Case i: G is connected.

Then $\chi(G) \geq 2$. Let V – D = {u_1, u2 …, u_k}. Since D is a dominating set, each u_i is adjacent to a vertex of D and hence, contributes at least one degree to D. Since $\chi(\langle D\rangle) \geq 2$, D contains at least one edge which contributes 2 degrees to D. Hence, $\sum_{u\in D} deg(u) \geq k + 2$, a contradiction.

Case ii: G is disconnected.

If G is totally disconnected, then V = D and hence, |V – D| = k = 0, a contradiction. Hence, G has a non trivial component and <D> contains at least one edge. Then by a similar argument as in case (i), a contradiction arises. Thus in both cases, we arrive at a contradiction, proving that k = 0, i.e., $|V - D| = \sum_{u\in D} deg(u) = 0$. Therefore, V = D and hence, for each $u \in V$, deg(u) = 0. Thus, G is a totally disconnected graph and hence, G = kK_1.

Corollary 2.10: For any non trivial connected graph with a dom-chromatic set D, $\sum_{u\in D} deg(u) \geq |V - D| + 2$.

Proof: If G is vertex-color-critical, then V = D and $\sum_{u\in D} deg(u) = 2q \geq 2 = |V - D| + 2$.

Suppose G is not vertex-color-critical, then since G is non trivial, $\chi(G) \geq 2$. Thus, by a similar argument as in Case (i) of proposition 2.9, $\sum_{u\in D} deg(u) \geq |V - D| + 2$.

Proposition 2.11: For any graph G, $\left\lfloor \frac{p}{\Delta(G)+1} \right\rfloor \leq \gamma_{ch}(G)$ and equality holds if and only if G = pK_1, $p \geq 1$.

Proof: From Theorem 2.2, lower bound is trivial. If G = pK_1, then the result follows. Suppose $\left\lfloor \frac{p}{\Delta(G)+1} \right\rfloor = \gamma_{ch}(G) = k$ and D is a γ_{ch}-set of G.

Case i: G is connected. If $k \geq 2$, then G is a non trivial connected graph. Then by corollary 2.10, $|V - D| < \sum_{u\in D} \deg(u)$. Thus, $p - k < \sum_{u\in D} \deg(u) \leq k\Delta(G)$ and hence, $\frac{p}{\Delta(G)+1} < k$.

Hence, $k > \frac{p}{\Delta(G)+1} \geq \left\lfloor \frac{p}{\Delta(G)+1} \right\rfloor = k$, a contradiction. Thus, k = 1 and hence, $\gamma_{ch}(G) = 1$. Therefore, G = K_1.

<u>Case ii:</u> G is disconnected.

Suppose G is not totally disconnected, then G has atleast one non trivial component. By a similar argument as in case (i), contradiction arises. Therefore, G = pK_1.

Proposition 2.12:

(i)If G is a connected graph, then $\gamma_{ch}(G) = p$ if and only if G is a vertex-color- critical or G is color-critical graph.

(ii)If G is a disconnected graph, then $\gamma_{ch}(G) = p$ if and only if either G is a null graph or has exactly one non trivial component, which is vertex-color-critical or color-critical.

3. Main result

In this section, bipartite graphs are studied. Since the dc-number of a bipartite graph is either $\gamma(G)$ or $\gamma(G) + 1$, graphs with exact bounds are identified. Further certain classes of graphs whose dc-number is half of their order are found with a given diameter.

Proposition 3.1: Let G be a forest with each component of diameter at most 4. Then

(i) If G has at least one component of diameter 3, then $\gamma_{ch}(G) = \gamma(G)$.

(ii) If G has at least one component of diameter 4 with its center adjacent to a pendant vertex, then $\gamma_{ch}(G) = \gamma(G)$.

(iii) If none of the components satisfy (i) or (ii), then $\gamma_{ch}(G) = \gamma(G) + 1$.

Theorem 3.2: If G is a bipartite graph with no isolated vertices, then $\gamma_{ch}(G) \le \frac{p}{2} + 1$ and $\gamma_{ch}(G) = \frac{p}{2} + 1$ if and only if $G = \frac{p}{2} K_2$.

Proof: Since the dc-number of a bipartite graph is either $\gamma(G)$ or $\gamma(G) + 1$, the upper bound follows. Also, if G is the union of independent edges, then equality holds. Conversely, suppose that $\gamma_{ch}(G) = \frac{p}{2} + 1$ and $\{V_1,V_2\}$ be the vertex partition of V.

<u>Claim 1:</u> $|V_1| = |V_2|$

Without loss of generality, let $|V_1| < |V_2|$. Then, $|V_1| < \frac{p}{2}$. Thus, $\gamma_{ch}(G) \le |V_1| + 1 < \frac{p}{2} + 1$, a contradiction. Hence $|V_1| = |V_2|$.

Let $S = V_1 \cup \{x\}$, $x \in V_2$. Then, S is a dom-chromatic set of G and $|S| = \frac{p}{2} + 1$. Hence, S is a γ_{ch}-set of G.

Claim 2: $|E(<S>)| = 1$

Suppose $|E(<S>)| \geq 2$, then x is adjacent to more than one vertex of V_1. Let $N_{<S>}(x) = \{x_1, x_2, \ldots, x_r\}$, $r \geq 2$. Then for each i, $1 \leq i \leq r$, $S - x_i$ induces a 2-chromatic graph. Since S is minimal, $S - x_i$ cannot be a dominating set of G. Thus, there exists a unique vertex $y_i \in V_2$ such that $x_i y_i \in E$. Similarly, as S is maximal, for each $z \in V_1 - N[x]$ there exists a unique vertex $z' \in V_2$ such that $zz' \in E$. Then, $|V_2| \geq \frac{p}{2} + 1$, a contradiction. Thus, each $x \in V_2$, x is adjacent to only one vertex of V_1. By claim 1, deg(x) = 1 for each $x \in V_1 \cup V_2$. Therefore, $G = mK_2$.

Corollary 3.3: If G is a bipartite graph with no isolated vertices and $\gamma_{ch}(G) = \frac{p}{2} + 1$, then $\gamma(G) = \frac{p}{2}$.

Proposition 3.4:

(i) A bipartite graph G has a dominating edge if and only if $\gamma_{ch}(G) = 2$

(ii) If G is a tree of diameter 2 or 3 then, $\gamma_{ch}(G) = 2$.

Proof:

(i) Let e = xy be a dominating edge of G. Then {x, y} is a γ_{ch}-set of G. Conversely suppose $\gamma_{ch}(G) = 2$ and S be any γ_{ch}-set of G, then $|S| = 2$. Since $\chi(<S>) = 2$, $<S> = K_2$. Further, S is also a dominating set of G implies G has a dominating edge.

(ii) Since diam(G) = 2 or 3, G has a dominating edge. Then by (i), $\gamma_{ch}(G) = 2$.

Theorem 3.5: If T is a tree with diam(G) ≤ 4, then $\gamma_{ch}(G) = \frac{p}{2}$ if and only if G is $K_{1,3}$, P_4 or a graph in the family given in figure 1.

Proof: Necessary condition is trivial. Suppose that $\gamma_{ch}(G) = \frac{p}{2}$. Clearly, diam(G) ≥ 2. For otherwise, $G = K_2$ and then $\gamma_{ch}(G) = p$, a contradiction.

Case i: diam(G) = 2.

Then $G = K_{1,n}$, $n \geq 2$, which implies $\gamma_{ch}(G) = 2$. Then p = 4. Therefore, $G = K_{1,3}$.

Case ii: diam(G) = 3.

From Proposition 3.5(ii), $\gamma_{ch}(G) = 2$. Therefore, p = 4 and hence, $G = P_4$.

Case iii: diam(G) = 4.

Then there exists a unique vertex x such that e(x) = 2. Therefore, N(x) can be partitioned into three sets as follows:

$S_1 = \{y \in N(x) \mid deg(y) = 1\}$

$S_2 = \{y \in N(x) \mid deg(y) = 2\}$

$S_3 = \{y \in N(x) \mid deg(y) \geq 3\}$

Let $|S_i| = m_i$, $1 \leq i \leq 3$.

Claim 1: $m_1 > 0$, $m_2 > 0$ and $m_3 > 0$ cannot be simultaneously hold

Suppose not. Then $\gamma_{ch}(G) = m_2 + m_3 + 1$ and $p \geq 1 + m_1 + 2m_2 + 3m_3$, a contradiction to $\gamma_{ch}(G) = \frac{p}{2}$.

As diam(G) = 4, at least one of m_2 or m_3 is not 0.

Claim 2: $m_1 > 0$, $m_2 = 0$ and $m_3 > 0$ cannot simultaneously hold.

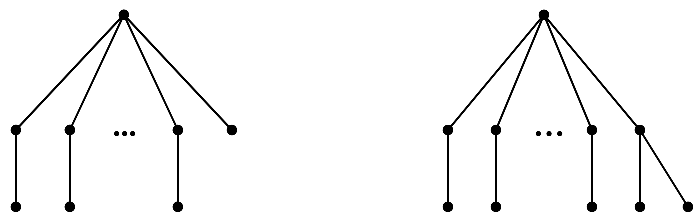
Suppose claim does not hold. Then $\gamma_{ch}(G) = m_3 + 1$ and $p \geq 3m_3 + m_1 + 1$, a contradiction.

Sub Case i: $m_1 > 0$

By claim 1, one of m_2 and m_3 is zero. By claim 2, m_2 cannot be zero. Hence, $m_2 > 0$ and $m_3 = 0$. Since diam(G) = 4, $m_2 \geq 2$ and $\gamma_{ch}(T) = m_2 + 1$. Hence, $p = 2m_2 + 2 = m_1 + 2m_2 + 1$. Therefore, $m_1 = 1$. Then G is in the family given in Figure 1(a).

Sub Case ii: $m_1 = 0$.

Suppose $m_2 = 0$. Then $m_3 \geq 2$ and $\gamma_{ch}(G) = m_3 + 1$. Therefore, $p = 2m_3 + 2$. Also $p \geq 3m_3 + 1$ implies $1 \geq m_3$, a contradiction. So let $m_3 = 0$. Then, $m_2 \geq 2$ and $p = 2m_2 + 1$. This implies G is a tree with odd number of vertices, a contradiction. Hence, both m_2 and m_3 are not zero. Then $\gamma_{ch}(G) = m_2 + m_3 + 1$ and hence, $p = 2m_2 + 2m_3 + 2$. Also $p \geq 2m_2 + 3m_3 + 1$ implies $1 \geq m_3$. Therefore, $m_3 = 1$ and hence, $\gamma_{ch}(G) = m_2 + 2$. Let $y \in S_3$. But, $p = 2m_2 + 4$ implies deg(y) = 3. Then G is a tree in the family given in Figure 1(b).



(a) (b)

Figure 1

Theorem 3.6: Suppose G is a forest with each component of diameter 1 or 2. Then $\gamma_{ch}(G) = \frac{p}{2}$ if and only if G is either $K_{1,3} \cup mK_2$, $m > 0$ or $2P_3 \cup mK_2$, $m \geq 0$.

Proof: Necessary condition can be easily verified. Conversely, suppose $\gamma_{ch}(G) = \frac{p}{2}$. Let $G_1, G_2 \ldots, G_r$, $r \geq 2$ be the components of G.

Claim 1: All components cannot be of diameter 1.

Suppose not. Then G = mK_2, $m \geq 1$. By Theorem 3.2, $\gamma_{ch}(G) = m + 1 = \frac{p}{2} + 1$, a contradiction.

Case i: All components are of diameter 2.

Then for each i, $1 \leq i \leq r$,

$G_i = K_{1,n_i}$ $n_i \geq 2$

$\Rightarrow$ $\gamma_{ch}(G) = r + 1$ ----------------- (1)

$\Rightarrow$ $p = 2r + 2$ ----------------- (2)

Claim 2: G cannot have K_{1,n_i} , $n_i \geq 4$ as a sub graph.

Suppose $G_1 = K_{1,n_1}$, $n_1 \geq 4$. From (1), $|V(G_i)| \geq 3$ for each i. Then, $p \geq 5 + 3(r-1) = 3r + 2$, a contradiction to (2).

Claim 3: G cannot have more than 2 components.

Suppose $r \geq 3$. Then $p \geq 3r \geq 2r + 3$, contradiction to (2).

As r = 2, from (2), p = 6. Since both components are of diameter 2, G = $2P_3$ and (ii) holds.

Case ii : G has components of diameter 1 and diameter 2.

Hence, each components is either a $K_{1,n}$, $n \geq 2$ or a K_2 . Let r_1, r_2 be the number of components of $K_{1,n}$ and K_2 respectively. Then $\gamma_{ch}(G) = r_1 + r_2 + 1$. Therefore,

$p = 2r_1 + 2r_2 + 2$ ----------------- (3)

Claim 4: $r_1 \leq 2$.

Suppose not. Then $p \geq 3r_1 + 2r_2 > 2r_1 + 2r_2 + 2$, a contradiction to (3).

Suppose $r_1 = 1$. Then $\gamma_{ch}(G) = r_2 + 2$ and $p = 2r_2 + 4$. Hence, $K_{1,n}$ can have only 4 vertices. Then $G = K_{1,3} \cup mK_2$, $m > 0$ and (i) holds. If $r_1 = 2$, then $\gamma_{ch}(G) = r_2 + 3$. Then $p = 2r_2 + 6$. Since two $K_{1,n}$'s with $n \geq 2$ and 6 vertices is $2P_3$, $G = 2P_3 \cup mK_2$ and (iii) holds.

Theorem 3.7. If G is a forest of even order without isolated vertices and each component is of diameter at most 3 with at least one component of diameter 3, then $\gamma_{ch}(G) = \frac{p}{2}$ if and only if G is

i. mP_4, $m > 1$ or

ii. $mP_4 \cup nK_2$, $m, n \geq 1$.

Proof: Necessary condition is easily verified. Suppose $\gamma_{ch}(G) = \frac{p}{2}$. Let r_1, r_2 and r_3 be number of components of diameter 1, 2 and 3 respectively. Then

$p \geq 2r_1 + 3r_2 + 4r_3$ ----------------- (1)

and $r_3 > 0$ ----------------- (2)

Claim 1: $r_1 = 0$ and $r_2 > 0$ cannot hold.

Suppose not. Then $\gamma_{ch}(G) = r_2 + 2r_3$ and hence, $p = 2r_2 + 4r_3$. From (1), $p \geq 4r_3 + 3r_2$, a contradiction.

Claim 2: $r_1 > 0$ and $r_2 > 0$ cannot hold.

Suppose not. Then $\gamma_{ch}(G) = r_1 + r_2 + 2r_3$, which implies $p = 2r_1 + 2r_2 + 4r_3$ a contradiction to (1).

From claim 1, claim 2 and from (2), only 2 cases are to be considered.

Case i: $r_1 = 0$ and $r_2 = 0$.

Therefore, $\gamma_{ch}(G) = 2r_3$, and hence, $p = 4r_3$. Since each component has at least 4 vertices, $G = r_3P_4$ and (i) is proved.

Case ii : $r_1 > 0$, $r_2 = 0$.

Then $\gamma_{ch}(G) = r_1 + 2r_3$, which implies $p = 2r_1 + 4r_3$ and hence, $G = r_3P_4 \cup r_1 K_2$ and (ii) holds.

Theorem 3.8. If G is a forest with isolated vertices and each non-trivial component is of diameter at most 2 with at least one component of diameter 2, then $\gamma_{ch}(G) = \frac{p}{2}$ if and only if $G = \bigcup_{i=1}^{i=k} K_{1,n_i} \cup rK_2 \cup (\sum_{i=1}^{k} n_i - k - 2)K_1$, $n_i \geq 2$ for each i, $k \geq 1$, $r \geq 0$, $\sum_{i=1}^{k} n_i - k - 2 > 0$.

Proof : Necessary condition is trivial. So suppose $\gamma_{ch}(G) = \frac{p}{2}$.

Case i : All non trivial components are of diameter 2.

Then, $G = \bigcup_{i=1}^{i=k} K_{1,n_i} \cup mK_1$, $n_i \geq 2$, which implies $\gamma_{ch}(G) = k + m + 1$. Therefore, $p = 2k + 2m + 2$. By the structure of G, $p = \sum_{i=1}^{k} n_i + k + m$. Therefore, $m = \sum_{i=1}^{k} n_i - k - 2$ and satisfies the given condition.

Case ii: G contains non trivial components of diameter 1 and 2.

Then $G = \bigcup_{i=1}^{i=k} K_{1,n_i} \cup rK_2 \cup mK_1$, $r > 0$, $m > 0$. Therefore, $\gamma_{ch}(G) = k + r + m + 1$ and hence, $p = 2k + 2r + 2m + 2$. From the structure of G, $p = \sum_{i=1}^{k} n_i + k + 2r + m$.

Therefore, m = $\sum_{i=1}^{k}$ n_i - k - 2 and the result follows.

Notation 3.9. P_4^i is the family of graphs obtained from P_4 by randomly joining i vertices to the intermediate vertices. It can be seen that any tree of diameter 3 is P_4^i, i ≥ 0.

Theorem 3.10. If G is a forest with isolated vertices and the non-trivial components are of diameter at most 3 with at least one component of diameter 3, then $\gamma_{ch}(G) = \frac{p}{2}$ if and only if G is one of the following graphs.

i) $\bigcup_{i=1}^{i=m} P_4^{l_i} \cup rK_2 \cup nK_1$, r ≥ 0, $\sum_{i=1}^{m} l_i = n$

ii) $\bigcup_{i=1}^{i=m} P_4^{l_i} \cup (\bigcup_{j=1}^{s} K_{1,n_j}) \cup rK_2 \cup nK_1$, r ≥ 0, $n_j \geq 2$ for each j and $\sum_{i=1}^{m} l_i + \sum_{j=1}^{s} n_j = s + n$.

Proof: Necessary condition is trivial. Suppose $\gamma_{ch}(G) = \frac{p}{2}$.

<u>Case i</u>: All non trivial components are of diameter 3.

Then G = $\bigcup_{i=1}^{i=m} P_4^{l_i} \cup nK_1$. Therefore, p = 4m + $\sum_{i=1}^{m} l_i$ + n. As $\gamma_{ch}(G)$ = 2m + n, p = 4m + 2n. Then $\sum_{i=1}^{m} l_i$ = n. Hence, G satisfies (i) with r = 0.

<u>Case ii:</u> Non trivial components are of diameter 3 and 1.

Then G = $\bigcup_{i=1}^{i=m} P_4^{l_i} \cup rK_2 \cup nK_1$. Therefore, p = 4m + $\sum_{i=1}^{m} l_i$ + 2r + n. As $\gamma_{ch}(G)$ = 2m + r + n, p = 4m + 2r + 2n. Then $\sum_{i=1}^{m} l_i$ = n and hence, G satisfies (i) with r > 0.

<u>Case iii:</u> Non trivial components are of diameter 3 and 2.

Then $G = \bigcup_{i=1}^{i=m} P_4^{l_i} \cup (\bigcup_{j=1}^{s} K_{1,n_j}) \cup nK_1$, $n_j \geq 2$. This implies that $p = 4m + \sum_{i=1}^{m} l_i + \sum_{j=1}^{s} (n_j + 1) + n$. But $\gamma_{ch}(G) = 2m + s + n$ implies $p = 4m + 2s + 2n$. Therefore, $\sum_{i=1}^{m} l_i + \sum_{j=1}^{s} n_j = s + n$. Then G satisfies (ii) with r = 0.

Case iv: Non trivial components are of diameter 3, 2 and 1.

Then, $G = \bigcup_{i=1}^{i=m} P_4^{l_i} \cup (\bigcup_{j=1}^{s} K_{1,n_j}) \cup rK_2 \cup nK_1$, $n_j \geq 2$ and hence, $p = 4m + \sum_{i=1}^{m} l_i + \sum_{j=1}^{s} (n_j + 1) + 2r + n$. Also, $\gamma_{ch}(G) = 2m + s + r + n$ implies $p = 4m + 2s + 2r + 2n$. This leads to $\sum_{i=1}^{m} l_i + \sum_{j=1}^{s} n_j = s + n$. Hence, G satisfies (ii) with $r > 0$.

Theorem 3.11. If G is bipartite, then $\gamma_{ch}(G) = p$ if and only if either $G = K_2$ or $G = K_2 \cup (p - 2)K_1$.

Proof : If G is connected, then $\gamma_{ch}(G) = p$ if and only if G is vertex-color-critical. Since the only bipartite vertex-color-critical graph is K_2, the result follows. Suppose G is disconnected. The result follows from Proposition 4.1.7(ii).

Theorem 3.12. Let G be a tree of diameter 5. If at least one of the central elements of G is adjacent to a pendant vertex, then $\gamma_{ch}(G) = \gamma(G)$, otherwise $\gamma_{ch}(G) = \gamma(G) + 1$.

Proof: As diam(G) = 5, G has two central elements and they are adjacent. Let them be x and y. Then they are adjacent. Let $S_1 = \{u \mid u \in N(x) - y\}$, $S_2 = \{u \mid u \in N(y) - x\}$, $S_3 = \{u \mid u \in S_1 \text{ and } d(u) > 1\}$ and $S_4 = \{u \mid u \in S_2 \text{ and } d(u) > 1\}$.

Case i: Both x and y are not adjacent to any pendant vertices.

Then $S_1 \cup S_2$ is a γ-set of G and $S_1 \cup S_2 \cup \{x\}$ a γ_{ch}-set of G. Therefore, $\gamma_{ch}(G) = \gamma(G) + 1$.

Case ii: x or y is adjacent to a pendant vertex not both.

Suppose x is adjacent to a pendant vertex. Then $S_2 \cup S_3 \cup \{x\}$ is a γ_{ch}-set as well as a γ-set of G. Therefore, $\gamma_{ch}(G) = \gamma(G)$.

Case iii: Both x and y are adjacent to pendant vertices.

Then $S_3 \cup S_4 \cup \{x, y\}$ is a γ_{ch}-set as well as a γ-set of G. Therefore, $\gamma_{ch}(G) = \gamma(G)$.

Theorem 3.13. Let G be a tree of diameter 5. Then $\gamma_{ch}(G) = \frac{p}{2}$ if and only of G is a graph in the family of trees given in figure 2.

Proof: Necessary condition is trivial. Suppose $\gamma_{ch}(G) = \frac{p}{2}$. Let x and y be the central elements of G. Let $S_1 = \{u \mid u \in N(x) - y\}$ and $S_2 = \{u \mid u \in N(y) - x\}$.

Case i: Both x and y are not adjacent to any pendant vertices.

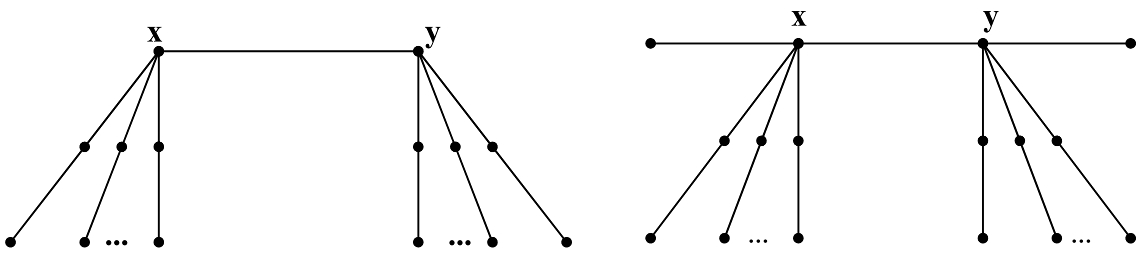
Then $S_1 \cup S_2 \cup \{x\}$ is γ_{ch}-set of G. Then $p = 2(|S_1| + |S_2| + 1)$, which implies that G is a tree in the family given in figure 3(a)

Case ii: At least one of x and y is adjacent to all pendant vertices

First the following claim is proved.

Claim :. Only one of x and y is adjacent to pendant vertices cannot hold.

Suppose not. Let x be adjacent to pendant vertices and y is not. Define $S_3 = \{u \mid u \in S_1$ and $d(u) > 1\}$. Then $S_2 \cup S_3 \cup \{x\}$ is a γ_{ch}-set of G. Therefore, $\gamma_{ch}(G) = |S_2| + |S_3| + 1$ and hence, $p = 2(|S_2| + |S_3| + 1)$. Also, from the structure of G, $p \geq 2(|S_2| + |S_3|) + 3$, a contradiction.



(a) **Figure 2** (b)

Hence, by the claim both x and y must be adjacent to pendant vertices. Let $S_4 = \{u \mid u \in S_2$ and $d(u) > 1\}$. Then $S_3 \cup S_4 \cup \{x, y\}$ is a γ_{ch}-set of G. Therefore, $\gamma_{ch}(G) = |S_3| + |S_4| + 2$ and hence, $p = 2(|S_3| + |S_4| + 2)$. Therefore, G is a tree in the family given in figure 2(b).

Proposition 3.14. If G is a bipartite graph of diameter 2, then $\gamma_{ch}(G) = 2$.

Proof : Let xyz be a diameteral path of G. Define 3 sets as follows:

$S_1 = N(x) - y$, $S_2 = N(y) - \{x, z\}$, $S_3 = N(z) - y$. Since G is bipartite, all the above 3 sets induce null graphs. If $S_1 = S_2 = S_3 = \phi$, then $G = K_{1,2}$ and hence, $\gamma_{ch}(G) = 2$. If $S_1 = S_3 = \phi$ and $S_2 \neq \phi$, again G is a $K_{1,n}$, $n \geq 3$. Therefore, $\gamma_{ch}(G) = 2$. So suppose S_1 or $S_3 \neq \phi$, say S_1 .

Claim: $S_1 = S_3$.

Let u ∈ S_1 - S_3. Since diam(G) = 2, d(u, z) = 2. As u is not adjacent to y, there exists a v such that uvz is a path. Then xyzvux is a 5-cycle, a contradiction.

Then {x, y} is a γ_{ch}-set of G.

Proposition 3.15. Let G be a tree of diameter 3. Then $\gamma_{ch}(G) = p - \Delta(G)$ if and only if G = P_4 or G is a tree in the family given in figure 3.

Proof : Clearly, $\gamma_{ch}(G) = 2$. Suppose, G = P_4. Then $\gamma_{ch}(G) = 2$, p = 4, $\Delta(G) = 2$ and the proposition holds. Suppose G is a tree given in Figure 4. Then $\gamma_{ch}(G) = 2$, p = deg(z) + 2 and $\Delta(G)$ = deg(z). Then the result holds.

Conversely, suppose $\gamma_{ch}(G) = p - \Delta(G)$. Let e = yz be a dominating edge. Then deg(y) ≥ 2 and deg(z) ≥ 2.

Claim : deg(y) ≥ 3 and deg(z) ≥ 3 cannot hold simultaneously.

Case i: deg(y) = deg(z) = 2

Then G = P_4 and the conditions are satisfied.

Case ii: deg(y) = 2 and deg(z) ≥ 3.

Let deg(z) = n ≥ 3. Then $\Delta(G)$ = n and p = n + 2. Therefore, $p - \Delta(G) = 2 = \gamma_{ch}(G)$ and G is a tree given in Figure 3.

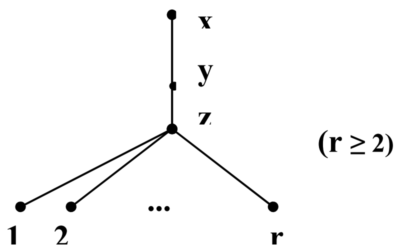


Figure 3

Proposition 3.16. Suppose G is a bipartite unicyclic graph and for each v ∈ V - V(C), d(v, C) = 1, then $\gamma_{ch}(G) \leq p - t$, where C is the unique cycle and t is the number of pendent vertices of G. Further the equality holds if and only if deg(v) > 2 for each v ∈ V(C).

Proof : From the given condition, it is clear that any vertex not in C is adjacent to exactly one vertex of C and is a pendant vertex. Then V(C) is a dom-chromatic set of G and |V(C)|= p – t. This gives the given upper bound. Suppose deg(v) >2 for each v ∈ V(C). Then V(C) is a γ_{ch}-set of G, and the bound is attained. Suppose $\gamma_{ch}(G) = p - t$. If there exists a vertex v in C of degree 2, then V(C) - v is a dom-chromatic set of G of cardinality p - t -1, a contradiction. Thus, deg(v) > 2 for each v ∈ V(C).

Proposition 3.17. If G is a path, then $\gamma_{ch}(G) = \frac{p}{2}$ if and only if G is P_4, P_6, or P_8.

Proof : Necessary condition is trivial. Conversely, suppose $\gamma_{ch}(G) = \frac{p}{2}$.

<u>**Case i:**</u> $p \equiv 0 \pmod 3$

From Proposition 4.1.5 (iv), $\gamma_{ch}(G) = \frac{p+3}{3}$. Then p = 6.

<u>**Case ii**</u>: $p \equiv 1 \pmod 3$

From Proposition 4.1.5(iv), $\gamma_{ch}(G) = \frac{p+2}{3}$. Then p = 4.

<u>**Case iii:**</u> $p \equiv 2 \pmod 3$

From Proposition 4.1.5(iv), $\gamma_{ch}(G) = \frac{p+4}{3}$. Then p = 8.

Proposition 3.18. If G is an even cycle, then $\gamma_{ch}(G) = \frac{p}{2}$ if and only if G is C_4, C_6, or C_8.

Theorem 3.19. If G is any (p, q) graph, q ≥ 1, then $\gamma_{ch}(G) = p - q + 1$ if and only if G contains exactly $\gamma_{ch}(G)$ -1 components and exactly one of the following holds.

i. each component is isomorphic to $K_{1,s}$'s, s ≥ 0

ii. exactly one component is a tree with diameter 3 or $K_{1,t}$, t ≥ 1 and every other component is isomorphic to $K_{1,m}$'s, m ≥ 0

Proof: Suppose G has $\gamma_{ch}(G)$ - 1 components with (i) or (ii) is satisfied. Let $\gamma_{ch}(G) - 1 = k$ and $G = G_1 \cup G_2 \cup .. \cup G_k$. Let each G_i be a (p_i, q_i)-graph. Then in both cases, $\gamma_{ch}(G) = k + 1$ and $q = \sum_{i=1}^{k} q_i = \sum_{i=1}^{k} (p_i - 1) = p - k$ and hence, $\gamma_{ch}(G) = p - q + 1$.

Conversely, suppose $\gamma_{ch}(G) = p - q + 1$. Suppose G has k components.

<u>**Claim 1:**</u> $k = \gamma_{ch}(G) - 1$.

Since q ≥ 1, $\chi(G) \geq 2$ and a γ_{ch}-set contains at least one vertex from each component,

$$k + 1 \leq \gamma_{ch}(G) \qquad \text{---------------------} \qquad (1)$$

Since each component is connected, q ≥ p - k, and thus, $k \geq p - q = \gamma_{ch}(G) - 1$. Therefore, $k + 1 \geq \gamma_{ch}(G)$. Therefore, from (1), $k + 1 = \gamma_{ch}(G)$.

Let $G_1, G_2, \ldots, G_k$ be the components of G. Without loss of generality, let $\gamma_{ch}(G_1) = \min_{1 \leq i \leq k} \{\gamma_{ch}(G_i) \mid \chi(G_i) = \chi(G)\}$. From Claim 1, $\gamma_{ch}(G) - 1 = k$.

<u>**Claim 2:**</u> $\gamma_{ch}(G_1) = 2$ and $\gamma_{ch}(G_i) = 1$, i ≥ 2.

Since G contains an edge, $\gamma_{ch}(G_1) \geq 2$. Suppose $\gamma_{ch}(G_1) \geq 3$. By Claim 1, $\sum_{i=2}^{k} \gamma(G_i) \geq k - 1 \geq \gamma_{ch}(G) - 2$ and therefore, $\gamma_{ch}(G) = \gamma_{ch}(G_1) + \sum_{i=2}^{k} \gamma(G_i) \geq \gamma_{ch}(G) + 1$, a contradiction.

Thus, $\gamma_{ch}(G_1) = 2$. Now $\gamma_{ch}(G) = 2 + \sum_{i=2}^{k} \gamma(G_i)$. Therefore, $\sum_{i=2}^{k} \gamma(G_i) = \gamma_{ch}(G) - 2 = k - 1$.

$\gamma(G_i) = 1$, for each i.

<u>**Claim3:**</u> Each G_i is a tree.

Suppose G_j contains a cycle. Then $q_j \geq p_j$ and $q_i \geq p_i - 1$ for each $i \neq j$. Now, $q = \sum_{i=1}^{k} q_i = q_j + \sum_{\substack{i=1 \\ i \neq j}}^{k} q_i \geq p_j + \sum_{\substack{i=1 \\ i \neq j}}^{k} (p_i - 1) = \sum_{i=1}^{k} p_i - (k - 1) = p - \gamma_{ch}(G) + 2$.

Thus, $\gamma_{ch}(G) \geq p - q + 2$, a contradiction.

Each G_i is a tree and $\gamma(G_i) = 1$ imply that $G_i = K_{1,s}$, $s \geq 0$ for each $i \neq 1$. Since $G_1 = K_{1,s}$, $s \geq 0$ and $\gamma_{ch}(G_1) = 2$ imply that either G_1 is a $K_{1,s}$, $s \geq 1$ or a tree with diameter 3.

Theorem 3.20. If T is a tree with diam(T) = 4 and k is the number of non pendant vertices of T, then $\gamma_{ch}(T) = k$.

Proof : Since diam(T) = 4, T has unique center. Let u be the center of T and $S = \{x \mid x \in N(u), \deg(x) \geq 2\}$. If u is adjacent to a pendant vertex, then $S \cup \{u\}$ is a γ_{ch}-set of T. Hence, $\gamma_{ch}(T)$ is the number of non pendant vertices of T and let it be k. If u is not adjacent to any pendant vertex, then again $S \cup \{u\}$ is a γ_{ch}-set of T and the result follows.

Proposition 3.21. If G is a tree of diameter 3, then $\gamma_{ch}(G) = \gamma_{ch}(\overline{G})$ if and only if $G = P_4$.

Proof : If G is P_4, then the result is trivial. Conversely suppose $\gamma_{ch}(G) = \gamma_{ch}(\overline{G})$. Let the dominating edge of G be e = uv. Let deg(u) = m and deg(v) = n. Also in $\overline{G}$, $(N(u) - v) \cup (N(v) - v)$ is a K_{m+n-2} and $\chi(\overline{G}) = m + n - 2$. Clearly, the above set is dominating $\overline{G}$. Hence, $\gamma_{ch}(\overline{G}) = m + n - 2$. Since $\gamma_{ch}(G) = \gamma_{ch}(\overline{G})$, m + n = 4 and the result follows.

Proposition 3.22. If G and $\overline{G}$, both bipartite with diameter 3, then $\gamma_{ch}(G) = \gamma_{ch}(\overline{G})$ if and only if G = P_4.

Solution: If G = P_4, then from Proposition 3.21, the equality holds. Conversely let the equality hold and uvwx be a diameteral path in G. Then N(v) and N(w) induce null graphs. Further, as $\overline{G}$ is bipartite both N(v) and N(w) are of cardinality 2.

<u>**Case i:**</u> both u and x are pendant vertices.

Then G = P_4 and hence, G is P_4.

<u>**ase ii:**</u> at least one of u and x is non pendant vertex.

Suppose u is a non pendant vertex and then by similar argument |N(u)| = 2 and N(u) is a null graph. Let N(u) = {v, u_1}. Clearly, u_1 is not adjacent to x, otherwise a 5-cycle is induced. Then

{ u_1, v, x} induces C_3, a contradiction.

Thus, from case i and ii, a solution is obtained only when G is P_4.

Theorem 3.23. If G is a tree of diameter 3, then $\gamma_{ch}(G) + \gamma_{ch}(\overline{G}) = p$.

Since G is a tree of diameter 3, G has a dominating edge. Therefore, $\gamma_{ch}(G) = 2$. Let uv be the dominating edge of G and V_1, V_2 be the set of pendant vertices adjacent to u and v respectively. Then in $\overline{G}$, $\langle V_1 \cup V_2 \rangle$ induces a complete graph K_{p-2} and, u and v are non adjacent. Further u is adjacent to each vertex of V_1 and v is adjacent to each vertex of V_2 in $\overline{G}$. Thus, $V_1 \cup V_2$ is a γ_{ch}-set of $\overline{G}$. Therefore, $\gamma_{ch}(\overline{G}) = p - 2$ and (i) follows.

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