International Journal of Engineering Science, Advanced Computing and Bio-Technology
Vol. 1, No. 2, April –June 2010, pp. 71-79

Acyclic Weak Convex Domination Critical Graphs

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Abstract: *In a graph G = (V, E), a set $D \subset V$ is a weak convex dominating(WCD) set if each vertex of V-D is adjacent to at least one vertex in D and $d_{<D>}(u,v) = d_G(u,v)$ for any two vertices u, v in D. A weak convex dominating set D, whose induced graph <D> has no cycle is called acyclic weak convex dominating(AWCD) set. The domination number $\gamma_{ac}(G)$ is the smallest order of a acyclic weak convex dominating set of G and the codomination number of G, written $\gamma_{ac}(\bar{G})$, is the acyclic weak convex domination number of its complement. In this paper we study the change in the behaviour of acyclic weak convex domination number with respect to addition of edges in the respective graph.*

Keywords: *domination number, distance, eccentricity, radius, diameter, self-centered, neighbourhood, weak convex dominating set, acyclic weak convex dominating set.*

1. Introduction

Graphs discussed in this paper are undirected and simple. Unless otherwise stated the graphs which we consider are connected graphs only. For a graph G, let V(G) and E(G) denote its vertex and edge set respectively. The *degree* of a vertex v in a graph G is denoted by $deg_G(v)$. The length of any shortest path between any two vertices u and v of a connected graph G is called the *distance between u and v* and is denoted by $d_G(u, v)$. The distance between two vertices in different components of a disconnected graph is defined to be ∞. For a connected graph G, the *eccentricity* $e_G(v) = \max\{d_G(u, v): u \in V(G)\}$. If there is no confusion, we simply use the notion deg(v), d(u, v) and e(v) to denote degree, distance and eccentricity respectively for the concerned graph. The minimum and maximum eccentricities are the *radius* and *diameter* of G, denoted r(G) and diam(G) respectively. When these two are equal, the graph is called *self-centered* graph with radius r, equivalently is *r self-centered*. A vertex u is said to be an *eccentric vertex* of v in a graph G, if d(u, v) = e(v). In general, u is called an eccentric vertex, if it is an eccentric vertex of some vertex. For $v \in V(G)$, the *neighbourhood* $N_G(v)$ of v is the set of all vertices adjacent to v in G. The set $N_G(v) = N_G(v) \cup \{v\}$ is called the *closed neighbourhood* of v. A set S of edges in a graph is said to be *independent* if no two of the edges in S are adjacent. An edge

Received: 08 February, 2009; Revised: 3 March, 2009; Accepted: 14 March, 2009

e = (u, v) is a *dominating edge* in a graph G if every vertex of G is adjacent to at least one of u and v.

The concept of domination in graphs was introduced by Ore. A set D$\subseteq$V(G) is called *dominating set* of G if every vertex in V(G)-D is adjacent to some vertex in D. D is said to be a *minimal* dominating set if D-{v} is not a dominating set for any $v\in$D. The *domination number* γ(G) of G is the minimum cardinality of a dominating set. We call a set of vertices a γ-set if it is a dominating set with cardinality γ(G). Different types of dominating sets have been studied by imposing conditions on the dominating sets. A dominating set D is called *connected* (*independent*) dominating set if the induced subgraph <D> is connected (independent). D is called a *total dominating set* if every vertex in V(G) is adjacent to some vertex in D.

A cycle D of a graph G is called a *dominating cycle* of G, if every vertex in V – D is adjacent to some vertex in D. A dominating set D of a graph G is called a *clique dominating set* of G if <D> is complete. A set D is called an *efficient dominating set* of G if every vertex in V – D is adjacent to exactly one vertex in D. A set D $\subseteq$ V is called a *global dominating set* if D is a dominating set in G and $\overline{G}$. A set D is called a *restrained dominating set* if every vertex in V(G)-D is adjacent to a vertex in D and another vertex in V(G)-D. A set D is a *weak convex dominating set* if each vertex of V-D is adjacent to at least one vertex in D and the distance between any two vertices u and v in the induced graph <D> is equal to that of those vertices u and v in G. By $\gamma_c, \gamma_i, \gamma_t, \gamma_o, \gamma_k, \gamma_e, \gamma_g, \gamma_r$ and γ_{wc}, we mean the minimum cardinality of a connected dominating set, independent dominating set, total dominating set, cycle dominating set, clique dominating set, efficient dominating set, global dominating set and restrained dominating set respectively.

In this paper we study the change in the behaviour of acyclic weak convex domination number with respect to addition of edges in the respective graph. In this paper we define a graph called k – Acyclic Weak Convex Domination critical graph and study the properties possessed by the graph with respect to k=2 and 3, and from which, we generalised some of the properties with respect to any value of k. Also we studied several interesting properties with respect to the diameter and radius of the graph.

2.Main Results

Definition 2.1:

A graph is said to be A.W.C.D critical graph, if for every edge e$\notin$E(G), γ_{ac}(G+e) < γ_{ac}(G).

If G is A.W.C.D critical graph with $\gamma_{ac}(G)$ = k, then G is said to be k – A.W.C.D critical graph.

Observations :

2.1 : A graph G is 1-critical $\Leftrightarrow$ G = K_p.

2.2 : If γ_{ac} = 2, then γ_{wc} = 2.

Theorem 2.1:

G is 2- A.W.C.D critical $\Leftrightarrow$ the following hold good;

(i) G is 2-domination; and

(ii) For any two non-adjacent vertices one of them is of degree (n-2).

Theorem 2.2:

For any 2-A.W.C.D graph, $\Delta \leq$ n - 2 and $\delta \leq$ n – 3.

Theorem 2.3:

Any 2-A.W.C.D critical graph has diameter which equals to two.

Proof:

Let x and y be any two non-adjacent vertices of G. Then in G + xy either {x} or {y} will form a dominating set. Without loss of generality, assume that {x} will form a generating set. Then all the neighbours of $N_1(y)$ will be adjacent with x in G+ xy and hence in G also. Therefore, diameter of G must be equal to 2 as it cannot be equal to one.

Theorem 2.4:

Any 2-A.W.C.D critical graph is a block.

Proof:

Let v be a cut vertex of G and v_1 be the pendent vertex joined with v. Let {u, v} be a dominating edge of G. Then there exists at least one vertex $u_1 \in N_1(u)$ such that $d(u_1,v_1) \geq 3$, contradictory to the above theorem 2.3. Hence G is a block.

Theorem 2.5:

There exists no graph G for which both G and $\overline{G}$ are 2-A.W.C.D critical.

Theorem 2.6:

The diameter of a 3 - A.W.C.D critical graph is at most 3.

Proof:

Let x and y be any two non-adjacent vertices of G. Now G+xy has an A.W.C.D set of cardinality 2. Let it be {x, z} (or {y, z}). If x dominates some vertex in $N_1(y)$, then

$d(x, y) \leq 2$, otherwise z must dominate all of $N_1(y)$ in $G+xy$ and hence in G also. This implies that $d(x, y) \leq 3$.

Theorem 2.7:

If u is a cut vertex of a 3-A.W.C.D critical graph G, then u is adjacent to a pendant vertex of G.

Corollary 2.1:

If G is a 3-A.W.C.D graph with $\delta \geq 2$, then G is a block.

Theorem 2.8:

Let G be a 3-A.W.C.D critical graph, then there cannot be any geodesic cycle of length ≥ 6.

Proof:

Let C = $c_1, c_2, \ldots, c_n$, where $n \geq 6$ be a geodesic cycle of length greater than equal to 6 in a 3.A.W.C.D critical graph G. Now join c_1 and c_3. Then $G+c_1c_3$ has a 2-A.W.C.D set Clearly $\{c_1, c_2\}$ cannot dominate $G+c_1c_3$, since they cannot dominate the vertex c_5. If suppose $\{c_1, x\}$ dominates $G+c_1c_3$. Then the vertex x must dominates c_3, c_4, c_5 in $G+c_1c_3$. This implies that x is adjacent to c_1, c_3, c_4, c_5 in G. This implies that $d(c_1, c_4) = 2$ in G. This implies that C is not a geodesic cycle of length greater than or equal to 6. Hence the proof.

Theorem 2.9:

Let G be a 3-A.W.C.D critical graph and v be cut vertex in G. Then there cannot be any dominating set D with $e(v)=1$ in <D>.

Proof:

Let D= $\{u, v, w\}$ be the A.W.C.D set of G, where $u, w \in N_1(v)$. Let x be a pendant vertex adjacent to v. Now join u and w. Then either $\{u, v\}$ or $\{v, w\}$ can only dominate $G+uw$. Without loss of generality assume that $\{u, v\}$ form a dominating set. But $\{u, v\}$ cannot dominate any of the vertices which are uniquely dominated by w. Hence $\{u, v\}$ cannot be a dominating set. Similarly, $\{v, w\}$ also cannot be a dominating set. Therefore, $\gamma_{ac}(G + uw) = \gamma_{ac}(G)$ only. This is a contradiction to G is 3-A.W.C.D critical. Thus, both u and w cannot be adjacent to v in <D>.

Proposition 2.1:

If G is a k- A.W.C.D critical graph, then no two pendant vertices of G have a common neighbour.

Corollary 2.2:

Any k- A.W.C.D critical graph has at most k-pendant vertices.

Theorem 2.10:

Any 3-A.W.C.D critical graph has at most two cut vertices.

Proof:

If it has three cut vertices, then the diameter of the graph is at least 4, which is a contradiction to the fact that the diameter of any 3-A.W.C.D set is at most 3.

Theorem 2.11:

Any 3-A.W.C.D critical graph has at most one cut vertex.

Theorem 2.12:

If G is a connected 3-A.W.C.D critical graph with a cut vertex, then V(G)= V(A) $\cup$ V(B) such that A= $\{v\}$ and B is a self-centred graph of diameter 2 and the vertex v of A is adjacent to exactly one vertex of B.

Theorem 2.13:

If S is an independent set of r vertices in the connected 3-A.W.C.D critical graph G, then all the vertices of S has degree greater than or equal to r – 1.

Theorem 2.14:

Diameter of a k-A.W.C.D critical graph is at most k.

Theorem 2.15:

Any k-A.W.C.D critical graph can have at most (k-1) number of pendant vertices.

Proof:

Let $p_1,p_2,\ldots p_n$ be the pendant vertices of a k-A.W.C.D critical graph G. Let $q_1,q_2,\ldots.q_k$ be the supports of $p_1,p_2,\ldots\ldots p_k$ respectively. Then clearly $\{q_1,q_2,\ldots q_k\}$ is the only A.W.C.D set of G. Consider any two adjacent vertices of this A.W.C.D set. Without loss of generality assume that p_1 and p_2 are adjacent vertices of G. Now join p_1 and p_2. Then clearly any vertex of A.W.C.D set of G+p_1p_2 contains either p_1 or p_2. We can assume an A.W.C.D set A which contains p_1. Surely A contains $q_3,q_4,\ldots q_k$ also, since still they are cut vertices of G+p_1p_2. As A is connected, the connection between p_1 and the other vertices $q_3,q_4,\ldots q_k$ can be made through q_1 only. Therefore, A must contains $p_1,q_1,q_3,\ldots,q_k$. This implies |A|$\geq$ k, which is a contradiction to the fact that G is k-A.W.C.D critical. Hence G cannot have more than (k-1) number of pendant vertices.

Theorem 2.16:

There exists no A.W.C.D critical tree.

Proof:

Let T be a tree and let u and v be the pendant vertices of T. Let u' and v' be the support of u and v respectively. Now in T+uv any dominating set contains either u, u' and the internal vertices other than v in T or v, v' and the internal vertices other than u in T or all the internal vertices of T. In all the cases γ_{ac} (T+uv) = γ_{ac}(T) only. Which is a contradiction to T is critical. Hence the proof.

Now we define acyclic weak convex domatic Number of Graphs and study the properties of graphs related to it.

Definition 3.2 :

A partition $\{V_1,V_2,\ldots V_n\}$ of the vertex set V is said to be an acyclic weak convex restrained domination partition if each of the V_i is an A.W.C.R.D set.

The cardinality of the minimum A.W.C.R.D partition of V(G) is said to be acyclic weak convex restrained domatic number of G and is denoted by $d_{ac}(G)$.

As acyclic W.C.D set is also a W.C.D set, the following theorems follows trivially:

Theorem 2.17:

For any graph G, $d_{ac}(G) \leq \delta + 1$.

Theorem 2.18:

$d_{ac}(G) = \delta + 1 \Leftrightarrow G = Kp$

Corollary 2.3:

For any graph $G \neq K_p$, $d_{ac}(G) \leq \delta$.

Proposition 2.2:

For any graph, $\gamma_{ac} \times d_{ac} \leq p$.

Theorem 2.19:

Let G be a graph and D be an A.W.C.D set. Then the radius of the induced graph <D> induced by D is at the lowest by r-1, where r is the radius of G.

Theorem 2.20:

For any graph on p vertices G with radius r, $d_{ac}(G) \leq [p/(2r-2)]$.

Theorem 2.21:

For any graph G on p vertices with diameter d, $d_{ac}(G) \leq [p/(d-1)]$.

Question:

For a given two integers d, n with d<n, is it possible to construct a graph with diameter d and A.W.C.D partition n?

Consider n paths of length d-1:

$P_1(p_{1,1},p_{1,2},\ldots, p_{1,d})$, $P_2(p_{2,1},p_{2,2},\ldots, p_{2,d})$, …, $P_n(p_{n,1},p_{n,2},\ldots, p_{n,d})$

For any fixed j $(1 \leq j \leq d)$ join $P_{i,j}$ to $P_{k,j}$ $(1 \leq i, k \leq n)$ and $i \neq k$.

Then clearly each path will form an A.W.C.D set. Also the above graph has diameter d. Hence the above graph has nd number of vertices.

Proposition 2.3:

If a graph G on p vertices contains m number vertices of degree n-1 then $d_{ac} \leq m + \lfloor p/2 \rfloor$.

Proposition 2.4:

If both G and $\overline{G}$ are self-centred of diameter 2, then $d_{ac} \leq \lfloor p/3 \rfloor$.

Proposition 2.5:

For any graph G on p vertices, $d_{ac}(G) + d_{ac}(\overline{G}) \leq p+1$.

Corollary 2.4:

Let G be a graph on p vertices, then $d_{ac}(G) + d_{ac}(\overline{G}) = p+1 \Leftrightarrow G = K_p$ or $\overline{K_p}$.

Corollary 2.5:

For any graph $G \neq K_p$ or $\overline{K_p}$, $d_{ac}(G) + d_{ac}(\overline{G}) \leq p - 1$.

Proposition 2.6:

For any cycle C_n, where $n \geq 5$, $d_{ac}(C_n) + d_{ac}(\overline{C_n}) \leq \lfloor n/2 \rfloor + 1$.

Theorem 2.23:

For any graph $\gamma_{ac} + d_{ac} \leq n + 1$.

Theorem 2.24:

For any graph $G \neq K_p$ or $\overline{K_p}$, $d_{ac}(G) \times d_{ac}(\overline{G}) \leq (p-1)^2/4$.

Theorem 2.25:

If $G \neq K_p$ is regular, then $d_{ac} = \delta \Leftrightarrow$ there exists a δ - edge A.W.C dominating partition for G.

Corollary 2.6:

If G is k-regular with $d_{ac}(G) = k$, then the number of edges $q = k(k-1)$.

Proof:

Let G be a k-regular graph with $d_{ac}(G) = k$.

From the above theorem, we have $|V(G)| = 2k$. Thus G is a graph on 2k vertices with degree k for all the vertices. This implies that $q=\frac{k(k-1)}{2}+\frac{k(k-1)}{2}=k(k-1)$.

Corollary 2.7:

If G is k-regular and $d_{ac}(G) = k$, then G is a self-centred graph of diameter 2.

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