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Eccentric Domination in Graphs

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Abstract: *A subset D of the vertex set V(G) of a graph G is said to be a dominating set if every vertex not in D is adjacent to at least one vertex in D. A dominating set D is said to be an eccentric dominating set if for every $v \in V-D$, there exists at least one eccentric point of v in D. The minimum of the cardinalities of the eccentric dominating sets of G is called the eccentric domination number $\gamma_{ed}(G)$ of G. In this paper, bounds for γ_{ed}, its exact value for some particular classes of graphs are found.*

Key words: *Eccentric dominating set, Eccentric dominating number.*

1.Introduction

Let G be a finite, simple, undirected (p, q) graph with vertex set V(G) and edge set E(G). For graph theoretic terminology refer to Harary [4], Buckley and Harary [1].

Definition 1.1 Let G be a connected graph and u be a vertex of G. The **eccentricity** e(v) of v is the distance to a vertex farthest from v. Thus, e(v) = max {d(u, v) : u $\in$ V}. The **radius** r(G) is the minimum eccentricity of the vertices, whereas the **diameter** diam(G) is the maximum eccentricity. For any connected graph G, r(G) $\leq$ diam(G) $\leq$ 2r(G). v is a central vertex if e(v) = r(G). The **center C(G)** is the set of all central vertices. The central subgraph < C(G) > of a graph G is the subgraph induced by the center. v is a peripheral vertex if e(v) = d(G). The periphery P(G) is the set of all peripheral vertices.

For a vertex v, each vertex at a distance e(v) from v is an **eccentric vertex**. **Eccentric set of a vertex v** is defined as E(v) = {u $\in$ V(G) / d(u, v) = e(v)}.

Definition 1.2 The **open neighborhood** N(u) of a vertex v is the set of all vertices adjacent to v in G. N[v] = N(v) $\cup$ {v} is called the **closed neighborhood** of v. For a vertex v $\in$ V(G), $N_i(u)$ = {u $\in$ V(G) : d(u, v) = i} is defined to be the **ith neighborhood** of v in G.

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Definition 1.3 Let G be a graph with at least one edge. The set of vertices of **line graph** of G denoted L(G) consists of the edges of G with two vertices of L(G) adjacent whenever the corresponding edges of G are adjacent. A graph G is a **line graph**, if it is isomorphic to the line graph L(H) of some graph H.

Definition 1.4 [6] A set $S \subseteq V$ is said to be a **dominating set** in G, if every vertex in $V-S$ is adjacent to some vertex in S. A dominating set D is an **independent dominating set**, if no two vertices in D are adjacent that is D is an independent set. A dominating set D is a **connected dominating set**, if $< D >$ is a connected subgraph of G. A set $D \subseteq V(G)$ is a **global dominating set**, if D is a dominating set in G and $\overline{G}$.

Definition 1.5 A partition of V(G) is called domatic if all of its classes are dominating sets in G. The maximum number of classes of an domatic partition of V(G) is called the **domatic number** of G and is denoted by $d_d(G)$.

The various domination parameters introduced till now find many applications in covering of entire graph by the different sets with each of which has some specified property. These concepts are helpful to find centrally located sets to cover entire graph in which they are defined. The concept of eccentric set of a node has application in the location of farthest set of a node of a graph and hence in this paper, we define new concept named eccentric domination and study the structural properties of graph using this concept.

2. Eccentric domination

In this we initiate the study of new domination, defined as below.

Definition: 2.1 A set $D \subseteq V(G)$ is an **eccentric dominating set** if D is a dominating set of G and for every $v \in V - D$, there exists at least one eccentric point of v in D.

If D is an eccentric dominating set, then every superset $D' \supseteq D$ is also an eccentric dominating set. But $D'' \subseteq D$ is not necessarily an eccentric dominating set.

An eccentric dominating set D is a **minimal eccentric dominating set** if no proper subset $D'' \subseteq D$ is an eccentric dominating set.

Definition: 2.2 The **eccentric domination number** $\gamma_{ed}(G)$ of a graph G equals the minimum cardinality of an eccentric dominating set. That is, $\gamma_{ed}(G) = \min |D|$, where the minimum is taken over D in D, where D is the set of all minimal eccentric dominating sets of G.

Obviously, $\gamma(G) \leq \gamma_{ed}(G)$.

In the following, we first characterize minimum eccentric dominating set of a graph.

Theorem:2.1

An eccentric dominating set D is a minimal eccentric dominating set if and only if for each vertex u $\in$ D, one of the following is true.

(i) u is an isolated vertex of D or u has no eccentric vertex in D.

(ii) There exists some u $\in$ V–D such that N(u) $\cap$ D = {u}

Proof:

Assume that D is a minimal eccentric dominating set of G. Then for every vertex u $\in$ D, D – {u} is not an eccentric dominating set. That is there exists some vertex v in (V – D) $\cup$ {u} which is not dominated by any vertex in D – {u} or there exists v in (V – D) $\cup$ {u} such that v has no eccentric point in D – {u}.

Case (i):

Suppose u = v , then u is an isolate of D or u has no eccentric vertex in D.

Case (ii):

Suppose v $\in$ V – D

(a) If v is not dominated by D – {u}, but is dominated by D, then v is adjacent to only u in D, that is N(v) $\cap$ D = {u}.

(b) Suppose v has no eccentric point in D – {u} but v has an eccentric point in D. Then u is the only eccentric point of v in D. that is E(v) $\cap$ D = {u}.

Conversely, suppose that D is an eccentric dominating set and for each u $\in$ D one of the conditions holds, we show that D is a minimal eccentric dominating set.

Suppose that D is not a minimal eccentric dominating set, that is, there exists a vertex u $\in$ D such that D – {u} is an eccentric dominating set. Hence, u is adjacent to at least one vertex v in D – {u} and u has an eccentric point in D – {u}.

Therefore, condition (i) does not hold.

Also, if D – {u} is an eccentric dominating set, every element x in V - D is adjacent to at least one vertex in D – {u} and x has an eccentric point in D – {u}.

Hence, condition (ii) does not hold. This is a contradiction to our assumption that for each u $\in$ D, one of the conditions holds. This proves the theorem.

Next, we define eccentric point set of the Graph G and eccentric number of G and establish the relation between the domination number, eccentric number and eccentric domination number of a graph.

Definition: 2.3 Eccentric point set of G:

Let $S \subseteq V(G)$. Then S is known as an eccentric point set of G if for every $v \in V-S$, S has at least one vertex u such that $u \in E(v)$.

An eccentric point set S of G is a minimal eccentric point set if no proper subset S' of S is an eccentric point set of G.

S is known as a minimum eccentric point set if S is an eccentric point set with minimum cardinality.

Let e(G) be the cardinality of a minimum eccentric point set of G. e(G) can be called as **eccentric number of G.**

Let D be a minimum dominating set of a graph G and S be a minimum eccentric point set of G. Clearly, $D \cup S$ is an eccentric dominating set of G. Hence, $\gamma(G)+e(G) \leq \gamma_{ed}(G) \leq (n/2) + e(G)$.

Note: This lower bound is sharp since for the tree $T = K_n + K_1 + K_1 + K_m$, $n, m \geq 2$, $\gamma_{ed}(T) = \gamma(T)+2$, where $e(T) = 2$ for any tree with radius ≥ 2.

The following observations are obvious.

Observation: 1 For any tree T with $|V(T)| \geq 3$, $\gamma_{ed}(T) \leq n - \Delta(T)+2$.

Observation: 2 If G is disconnected, $\gamma(G) = \gamma_{ed}(G)$ since vertices from different components are eccentric to each other.

Observation: 3 $1 \leq \gamma_{ed}(G) \leq n$.

The bounds are sharp, since $\gamma_{ed}(G) = 1$ if and only if $G = K_n$ and $\gamma_{ed}(G) = n$ if and only if $G = \overline{K_n}$.

Observation: 4 $\gamma_{ed}(K_{,n}) = 1$.

The eccentric domination number of some standard classes of graphs are given in the following theorem.

Theorem: 2.2

(i) $\gamma_{ed}(K_{1,n}) = 2$, $n \geq 2$.

(ii) $\gamma_{ed}(K_{m,n}) = 2$.

(iii) $\gamma_{ed}(W_3) = 1$, $\gamma_{ed}(W_4) = 2$, $\gamma_{ed}(W_n) = 3$, for $n = 5$, $\gamma_{ed}(W_6) = 2$, $\gamma_{ed}(W_n) = 3$ for $n \geq 7$.

Proof:

(i) When $G = K_n$, radius = diameter = 1.

Hence any vertex $u \in V(G)$ dominate other vertices and is also an eccentric point of other vertices. Hence, $\gamma_{ed}(K_n) = 1$.

(ii) $G = K_{1,n}$. Let $D = \{u, v\}$, v-central vertex. The central vertex dominate all vertices in $V - D$ and u is an eccentric point of vertices of $V - D$. Hence, $\gamma_{ed}(K_{1,n}) = 2, n \geq 2$

(iii) $G = K_{m,n}$. $V(G) = V_1 \cup V_2$. $|V_1| = m$ and $|V_2| = n$ such that each element of V_1 is adjacent to every vertex of V_2 and vice versa.

Let $D = \{u, v\}$, $u \in V_1$ and $v \in V_2$. u dominate all the vertices of V_2 and it is eccentric to elements of $V_1 - \{u\}$. Similarly, v dominates all the vertices of V_1 and it is eccentric to elements of $V_2 - \{v\}$. Hence, D is a minimum eccentric dominating set and hence $\gamma_{ed}(K_{m,n}) = 2$.

(iv) $G = W_3 = K_4$. Hence, $\gamma_{ed}(W_3) = 1$

When $G = W_4$, Consider $D = \{u, v\}$, where u and v are adjacent non central vertices. D is a minimum eccentric dominating set. Therefore, $\gamma_{ed}(W_4) = 2$.

When $G = W_n$, let $D = \{u, v, w\}$ where u and v are any two adjacent non central vertices and w is the central vertex.. Then D is a minimum eccentric dominating set of G. Therefore, $\gamma_{ed}(W_n) = 3$, $n \geq 5$.

The next theorem gives exact value for eccentric domination number of graph obtained from deletion of a perfect matching (linear factor) from a complete graph on even number of vertices.

Theorem: 2.3

Let n be an even integer. Let G be obtained from the complete graph K_n by deleting edges of a linear factor. Then $\gamma_{ed}(G) = n/2$.

Proof:

Let u and v be a pair of non adjacent vertices in G. Then u and v are eccentric to each other. Also, G is unique eccentric point graph. Therefore, $\gamma_{ed}(G) \geq n/2$.

Consider $D \subseteq V(G)$ such that $\langle D \rangle = K_{n/2}$. D contains n/2 vertices such that each vertex in V–D is adjacent to at least one element in D and each element in V–D has its eccentric point in D. Hence $\gamma_{ed}(G) \leq n/2$. From (1) and (2) $\gamma_{ed}(G) = n/2$.

Following theorems give upper bound for eccentric domination number of a graph.

Theorem: 2.4

If G is of radius one and diameter two, then $\gamma_{ed}(G) \leq (n-t+2)/2$ where t is the number of vertices with eccentricity one.

Proof: Let $u \in V(G)$ such that $e(u) = 1$. Let t be the number of vertices with eccentricity one. u dominates all other vertices and for t–1 other vertices u is an eccentric point. Consider the remaining (n–t) vertices of G. They are also dominated by u but their eccentric points are different from u.

Hence, $\gamma_{ed}(G) \leq 1+(n-t)/2 = (n-t+2)/2$.

Theorem: 2.5

If G is of diameter two $\gamma_{ed}(G) \leq 1+\delta(G)$.

Proof: diam(G) = 2. Let $u \in V(G)$ such that $deg_G u = \delta(G)$. Consider, $D = \{u\} \cup N(u)$. This is an eccentric dominating set for G. Therefore, $\gamma_{ed}(G) \leq \delta(G)+1$ and D is a connected eccentric dominating set.

Corollary: 2.5

If G is self centered of diameter 2, then $\gamma_{ed}(G) + \gamma_{ed}(\overline{G}) \leq n + \delta - \Delta + 1$.

Proof: By Theorem 2.5, $\gamma_{ed}(G) \leq 1 + \delta$ and $\gamma_{ed}(\overline{G}) \leq 1+\delta(\overline{G}) = 1 + \overline{\delta}$. Hence $\gamma_{ed}(G) + \gamma_{ed}(\overline{G}) \leq 1 + \delta + 1 + \overline{\delta} = 2 + \delta + (n - 1 - \Delta) = n + \delta - \Delta + 1$.

If there exists no x in $N_2(u)$ such that x is adjacent to all vertices of N(u) and there exists no vertex y in N(u) which is adjacent to all vertices of $N_2(u)$ then N(u) is eccentric dominating set of G and $N_2(u)$ is an dominating set of $\overline{G}$. Thus, $\gamma_{ed}(G) + \gamma_{ed}(\overline{G}) \leq n + \delta - \Delta - 1$.

Theorem: 2.6

If G is of radius two and diameter three, then $\gamma_{ed}(G) \leq \min \{(n + deg_G u - 1)/2\}$, where the minimum is taken over all central vertices.

Proof: Let u be a central vertex with minimum degree. Consider N(u). N(u) dominates all the vertices of G and all the vertices in $N_2(u)$ are eccentric to u. Let S be a subset of $N_2(u)$ with minimum cardinality such that vertices in $N_2(u) - S$ has their eccentric vertices in S. Then $|S| \leq |N_2(u)|/2 = (n - deg_G u - 1)/2$. Now $N(u) \cup S$ is an eccentric dominating set of G. Hence, $\gamma_{ed}(G) \leq deg_G u + (n - deg_G u - 1)/2 = (n + deg_G u - 1)/2$. This proves the theorem.

Theorem: 2.7

If G is of radius two and diameter three, then $\gamma_{ed}(G) \leq \min \{n - deg_G u/2\}$, where the minimum is taken over all central vertices.

Proof: Let u be a central vertex with maximum degree. Then $V - N(u)$ is a dominating set of G. Vertices of N(u) are dominated by u, but vertices of N(u) may have their eccentric vertices in N(u) also. Let S be a subset of N(u) with minimum cardinality such that vertices in N(u) − S has their eccentric vertices in S. Then $|S| \leq deg_G u/2$. Now (V−N(u)) ∪ S is an eccentric dominating set of G. Hence, $\gamma_{ed}(G) \leq deg_G u/2 + (n - deg_G u) = n - deg_G u/2$. Hence, $\gamma_{ed}(G) \leq \min \{n - deg_G u/2\}$.

Corollary: 2.7.1

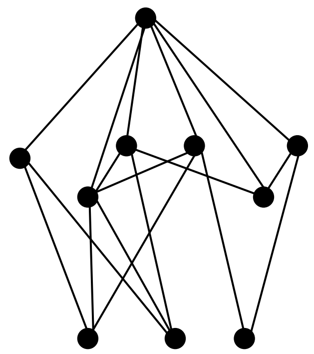
$\gamma_{ed}(G) \leq \min \{n - deg_G u/2, (n + deg_G u - 1)/2\}$, where the minimum is taken over all central vertices.

Corollary: 2.7.2

If G is of radius two and diameter three and if G has a pendent vertex v of eccentricity 3 then $\gamma_{ed}(G) \leq \Delta(G)$.

Proof: If G has a pendent vertex v of eccentricity 3 then its support u is of eccentricity 2. In this case N(u) is an eccentric dominating set. Thus, $\gamma_{ed}(G) \leq deg_G u \leq \Delta(G)$.

Example: Consider the following graph G.



By the theorem, $\gamma_{ed}(G) \leq \min \{n - deg_G u/2, (n + deg_G u - 1)/2\} = \min\{6,7,8\} = 6$.

Theorem :2.8

If G is of radius 2 with a unique central vertex u then $\gamma_{ed}(G) \leq n - deg(u)$.

Proof: If G is of radius 2 with a unique central vertex u then u dominates N[u] and the vertices in V − N[u] dominate themselves and each vertex in N(u) has eccentric vertices in V − N[u] only. Therefore, V − N(u) is an eccentric dominating set of cardinality n − deg(u), so that $\gamma_{ed}(G) \leq n - deg(u)$.

Corollary: 2.8

(i) If G is a unicentral tree of radius 2, then $\gamma_{ed}(G) \leq n - deg(u)$, where u is the central vertex.

(ii) If G is a bicentral tree of radius 2, then $\gamma_{ed}(G) \leq n - \deg(u)+1$, where u is a central vertex.

Theorem: 2.9

If G is of radius greater than two, then $\gamma_{ed}(G) \leq n - \Delta(G)$.

Proof: Let u be a vertex of maximum degree $\Delta(G)$. Then u dominates N[u] and the vertices in V − N[u] dominate themselves. Also, since diam(G) > 2, each vertex in N(u) has eccentric vertices in V − N[u] only. Therefore, V − N(u) is an eccentric dominating set of cardinality $n - \Delta(G)$, so that $\gamma_{ed}(G) \leq n - \Delta(G)$.

In the following three theorems, we analyze the bounds of eccentric domination number of a tree in terms of its domination number.

Theorem: 2.10

For a tree T, $\gamma(T) \leq \gamma_{ed}(T) \leq \gamma(T)+2$.

Proof:

Obviously, $\gamma(T) \leq \gamma_{ed}(T)$. Let d be the diameter of T. Let u, v ∈ V(T) such that e(u) = e(v) = d and d(u, v) = d. Then for any w ∈ V(T) either u or v is an eccentric point. Let D be any γ-dominating set of T. Then D ∪ {u, v} is an eccentric dominating set of T. Hence, $\gamma_{ed}(T) \leq \gamma(T)+2$.

The next theorem gives an upper bound for eccentric domination number of a tree in terms of its maximum degree.

Theorem:2.11

For a tree T, $\gamma_{ed}(T) \leq n - \Delta(T) + 1$.

Proof: If T has a vertex u of maximum degree which is not a support, then V − N(u) is an eccentric dominating set of T. If T has a vertex u of maximum degree which is a support of a pendent vertex v, then V − [N(u) − v] is an eccentric dominating set of T. Hence the theorem follows.

Theorem:2.12

For a tree T with radius greater than two, $\gamma_{ed}(T) < n - \Delta(T)$.

Proof: As in theorem 2.9, V − N(u) is an eccentric dominating set of cardinality $n - \Delta(G)$. Since the radius of T is atleast three, diameter of T is atleast 5. Consider a diametral path P. This path contains atleast six vertices and includes atmost two edges from the subgraph induced by N[u], that is it contains atmost three vertices from N[u].

Case (i): All vertices of P − N[u] (except pendent vertices) are support of some pendent vertices

In this case, we have to include all the vertices of P in a γ_{ed} set, but we can leave those pendent vertices from V − N(u) to form a γ_{ed} set. Therefore, $\gamma_{ed}(T) < n - \Delta(T)$.

Case (ii): Atleast one vertex of P − N[u] (except pendent vertices) is not a support

In this case, we can leave that vertex from V − N(u) to form a γ_{ed} set. Therefore, $\gamma_{ed}(T) < n - \Delta(T)$.

Hence atleast one vertex can be deleted from V − N(u) to form a minimal eccentric dominating set. Hence $\gamma_{ed}(T) < n - \Delta(T)$.

Theorem: 2.13

Let T be a tree with diameter $d > 2$. $\gamma(T) = \gamma_{ed}(T)$ if and only if T has a $\gamma(T)$ dominating set D containing at least two (pendent) peripheral vertices at distance d to each other.

Proof:

Assume $\gamma(T) = \gamma_{ed}(T)$. Let D be an eccentric dominating set with cardinality $\gamma(T) = \gamma_{ed}(T)$. Since in a tree, eccentric vertex of any vertex is a peripheral vertex D contains at least two peripheral vertices at distance d to each other.

On the other hand, assume that D is a $\gamma(T)$ dominating set of T containing at least two peripheral vertices u, v at distance d to each other. Every vertex $x \in V(T)$ has either u or v as an eccentric vertex. Hence D is also an eccentric dominating set of T. Hence, $\gamma(T) = \gamma_{ed}(T)$.

Example: (i) $\gamma(P_4) = 2 = \gamma_{ed}(P_4)$

(ii) $\gamma(K_{1,n}) = 1$, $\gamma_{ed}(K_{1,n}) = 2 = \gamma(K_{1,n})+1$

(iii) If $G = K_n + K_1 + K_1 + K_m$, n, m ≥ 2, $\gamma_{ed}(G) = 4 = \gamma(G) + 2$.

Theorem: 2.14 For a bicentral tree T with radius 2, $\gamma_{ed}(T) \leq \min\{n - \Delta(T)+1, 4\}$

Proof: Let u and v be the central vertices of T, Then N[u] and N[v] are eccentric dominating sets of T. V − [N(u) − v], V − [N(v) − u] are also eccentric dominating sets of T. Also deg u + deg v = n. Hence, $\gamma_{ed}(T) \leq n - \Delta(T)+ 1$. All the four vertices of a diametral path also form an eccentric dominating set. Hence the theorem follows.

Corollary:2.14 (i) For a bicentral tree T with radius 2, $\gamma_{ed}(T) = 2$ if and only if $T = P_4$.

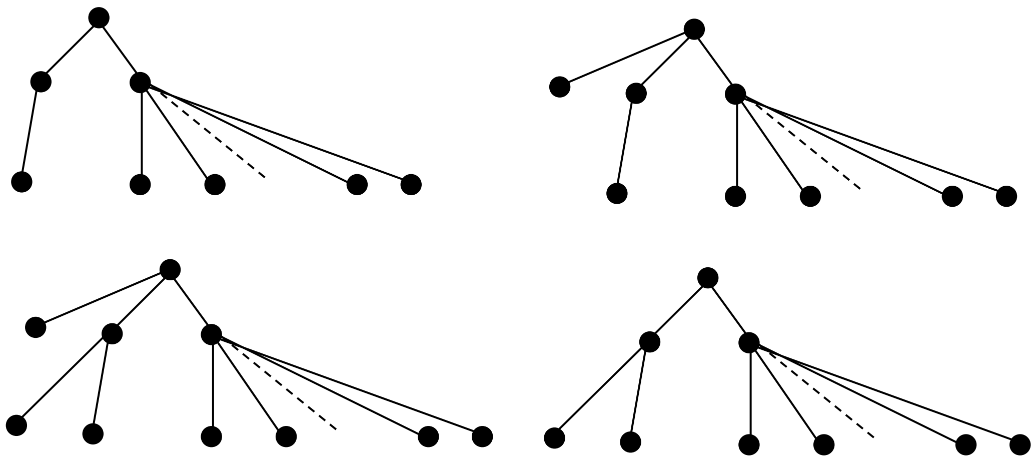
(ii) For a bicentral tree $T \neq P_4$ with radius 2, $\gamma_{ed}(T) = 3 = n - \Delta(T)+1$ if and only if T is a wounded spider having atmost one non wounded leg.

(iii) For a bicentral tree T with radius 2, $\gamma_{ed}(T) = 4$ if and only if degree of the central vertices are ≥ 3.

Theorem: 2.14 If T is a wounded spider having atmost one non wounded leg, $\gamma_{ed}(T) = n - \Delta(T) + 1$.

Proof: Proof is obvious.

Theorem: 2.15 Let T be a tree with radius 2 and diameter 4. $\gamma_{ed}(T) = n - \Delta(T)$ if and only if any one of the following is true: (i) $T = P_5$. (ii) T is a wounded spider having at least two non wounded legs. (iii) T is any one of the following four types of trees.



Proof: When T is a wounded spider having at least two non wounded legs or any one of the trees given above, it is clear that $\gamma_{ed}(T) = n - \Delta(T)$.

On the other hand assume that $\gamma_{ed}(T) = n - \Delta(T)$. Since T is a tree with radius 2 and diameter 4, T has a unique centre v.

Case (i): Let deg v = $\Delta(T)$.

Consider V – N(v). N(v) and V – N(v) are independent sets. N(v) is not an eccentric dominating set, since v has no eccentric vertex in N(v). Also each vertex in N(v) has either 0 or 1 neighbor in V – N(v). Hence T is a wounded spider having at least two non wounded legs.

Case (ii): Let deg v $\neq \Delta(T)$.

Let u be a vertex of maximum degree. It cannot be a pendent vertex. Therefore eccentricity of u is three. V – N(u) is an eccentric dominating set implies V – N(u) contains atleast one peripheral vertex (which is eccentric to u).

Now, consider the branches of T taking v as a root. If T contains a third branch with a peripheral vertex then V – N(u) is not a minimum eccentric dominating set. Therefore third branch must not contain any vertex of eccentricity 4. Suppose T contains more than three branches then also V – N(u) is not a minimum eccentric dominating set. Hence degree of v must be 2 or 3, with one branch containing u, second branch containing vertices of eccentricity 4 and the third one having a vertex of eccentricity 3 as end vertex or not. If the second branch contains more than two peripheral vertices then also V – N(u) is not an minimum eccentric dominating set. Hence T must be any one of the given four types of trees.

Next we find the exact value of eccentric domination number of a path.

Remark: 2.1 Let D be an eccentric dominating set of a path. Then any one of the following is true.

(i) D is a dominating set containing at least two peripheral nodes at distanced d to each other. That is D contains end vertices of the path.

(ii) D is a dominating set containing one peripheral vertex v and all vertices lying on the shortest path from v to a central node.

Remark: 2.2

An eccentric dominating set D of a path contains minimum number of vertices only when D contains two peripheral vertices at distance d (= diameter) to each other. That is D contains end vertices of the path.

Theorem: 2.16 $\gamma_{ed}(P_n) = \lceil n/3 \rceil$, if n = 3k+1,

$\gamma_{ed}(P_n) = \lceil n/3 \rceil + 1$, if n = 3k or 3k+2.

Proof: Case (i) n = 3k

An eccentric dominating set of P_n must contain the two end vertices.

Let $v_1, v_2, v_3, v_4, \ldots, v_{3k}$ represent the path P_n. D = $\{v_2, v_5, v_8, \ldots, v_{3k-1}\}$ is the only γ-dominating set of P_n. D is not an eccentric dominating set.

$D' = \{v_1, v_4, v_7, \ldots, v_{3k-2}, v_{3k}\}$ is an eccentric minimum dominating set and $|D'| = k+1 = \gamma(P_n) + 1$. Hence $\gamma_{ed}(P_{3k}) = \gamma(P_{3k})+1 = \lceil n/3 \rceil + 1$.

Case (ii) n = 3k+1

D = $\{v_1, v_4, v_7, \ldots, v_{3k-2}, v_{3k+1}\}$ is the minimum dominating set in P_n. It contains the two end vertices. Hence it is also an eccentric dominating set.

Hence $\gamma_{ed}(P_n) = \gamma(P_n) = \lceil n/3 \rceil$.

Case (iii) n = 3k+2

$D = \{v_2, v_5, v_8, \ldots, v_{3k+2}\}$ is a minimum dominating set. It contains one end vertex v_{3k+2} and it is not an eccentric dominating set. (other minimum dominating sets are also not eccentric). Hence $D \cup \{v_1\}$ is a minimum eccentric dominating set. Therefore, $\gamma_{ed}(P_n) = \gamma(P_n)+1 = \lceil n/3 \rceil +1$.

Following two theorems give the exact value of the eccentric domination number of cycles and their complement graphs.

Theorem: 2.17 (i) $\gamma_{ed}(C_n) = n/2$ if n is even.

(ii) $\gamma_{ed}(C_n) = \lceil n/3 \rceil$ or $\lceil n/3 \rceil +1$, if n is odd.

Proof of (i):

If n = 4, any two adjacent vertices of C_4 is an eccentric dominating set of C_4.

Hence $\gamma_{ed}(C_4) = 2$.

Let n = 2k and k > 2.

Let the cycle C_n be $v_1\, v_2\, v_3 \ldots v_{2k} v_1$. Each vertex of C_n has exactly one eccentric vertex (that is C_n is unique eccentric point graph).

Hence $\gamma_{ed}(C_n) \geq n/2$. ------------------- (1)

case(i) k-odd.

Consider $D = \{v_1, v_3, \ldots, v_k, v_{k+2}, \ldots, v_{2k-1}\}$. This D is an eccentric dominating set for C_n since D dominates C_n and v_i is an eccentric point of v_{i+k}.

Hence $\gamma_{ed}(C_n) \leq n/2$. ----------------------------------- (2)

From (1) and (2) $\gamma_{ed}(C_n) = n/2$.

case(ii) k even.

Let $D = \{v_1, v_3, \ldots, v_{k-1}, v_{k+2}, \ldots, v_{2k}\}$. This D is an eccentric dominating set for C_n, since D dominates C_n and v_i is an eccentric point of v_{i+k}.

Hence $\gamma_{ed}(C_n) \leq n/2$. ------------------------------- (3)

From (1) and (3) $\gamma_{ed}(C_n) = n/2$.

Proof of (ii):

When n is odd, each vertex of C_n has exactly two eccentric vertices.

If n = 2k+1, $v_i \in V(G)$ has v_{i+k}, v_{i+k+1} as eccentric points.

case(i) n = 3m, n odd $\Rightarrow$ m odd

$n = 3m = 2k+1 \Rightarrow 2k$ even and $2k = 3m-1$

$2k = 3(m-1)+2$

$k = (3(m-1)+2)/2 \Rightarrow k = 3l+1$ (since $m-1$ is even)

Consider $D = \{v_1, v_4, v_7, \ldots, v_k, v_{k+3}, \ldots, v_{2k-1}\}$. D is an eccentric dominating set and D is a γ-dominating set of C_n and $|D| = n/3 = m$

$$\text{Hence, } \gamma_{ed}(C_n) = n/3 = \gamma(C_n).$$

case(ii) $n = 3m+1$, n odd $\Rightarrow$ m is even.

Also $3m = 2k \Rightarrow$ k is a multiple of 3.

Consider $D = \{v_1, v_4, \ldots, v_{k+1}, v_{k+3}, v_{k+6}, \ldots, v_{2k-1}\}$. D is an eccentric dominating set and $|D| = \lceil n/3 \rceil = \gamma(C_n)$. Hence $\gamma_{ed}(C_n) = \lceil n/3 \rceil = m+1$.

case(ii) $n = 3m+2 \Rightarrow 3m$ is odd $\Rightarrow$ m is odd.

$$2k = 3m+1 = 3(m-1) + 4$$
$$k = 3l + 2$$

Consider $D = \{v_1, v_4, \ldots, v_{k-1}, v_k, v_{k+3}, \ldots, v_{2k+1}\}$. D is an eccentric dominating set with $\lceil n/3 \rceil + 1$ vertices and no γ-dominating set of C_n is an eccentric dominating set of C_n.

$$\text{Hence } \gamma_{ed}(C_n) = \lceil n/3 \rceil + 1.$$

Theorem: 2.18 $\gamma_{ed}(\overline{C_4}) = 2$, $\gamma_{ed}(\overline{C_5}) = 3$ and $\gamma_{ed}(\overline{C_n}) = \lceil n/3 \rceil$, $n \geq 6$.

Proof: Clearly, $\gamma_{ed}(\overline{C_4}) = 2$, $\gamma_{ed}(\overline{C_5}) = 3$, by Observation 2 and by Theorem 2.15.

Now, assume that $n \geq 6$. Let $v_1, v_2, v_3, \ldots, v_n, v_1$ form C_n. Then $\overline{C_n} = K_n - C_n$ and each vertex v_i is adjacent to all other vertices except v_{i-1}, and v_{i+1} in $\overline{C_n}$. Hence eccentric point of v_i in $\overline{C_n}$ is v_{i-1}, and v_{i+1} only. Hence any eccentric dominating set must contain either v_i or any one of v_{i-1}, v_{i+1}. So, $\gamma_{ed}(\overline{C_n}) \geq \lceil n/3 \rceil$, Now, we can consider an eccentric (minimal) dominating set as follows.

$\{v_1, v_4, v_7, \ldots, v_{3m-2}\}$ if $n = 3m$;

$\{v_1, v_4, v_7, \ldots, v_{3m+1}\}$ if $n = 3m+1$;

$\{v_1, v_4, v_7, \ldots, v_{3m+1}, v_{3m+2}\}$ if $n = 3m+2$;

Hence $\gamma_{ed}(\overline{C_n}) \leq \lceil n/3 \rceil$. Thus, it follows that $\gamma_{ed}(\overline{C_n}) = \lceil n/3 \rceil$, for $n \geq 6$.

Parthasarathy and Nandakumar introduced the concept of eccentricity preserving spanning trees of a given graph and its structural properties studied by them and also by Janakiraman [5]. Next theorem gives bound for the introduced parameter for the graphs having eccentricity preserving spanning trees.

Theorem: 2.19 If G has an eccentricity preserving spanning tree then $\gamma_{ed}(G) \leq \gamma(G) + 2$.

Proof: If G has an eccentricity preserving spanning tree, then the minimum number of vertices which are eccentric points of other vertices are 2.

$$\text{Hence, } \gamma_{ed}(G) \leq \gamma(G) + 2.$$

Next we give two results relating global domination number and eccentric domination number of a graph.

Observation: 2.4 If $\gamma(G) \geq 3$, γ_{ed} dominating set D of G is also a dominating set of $\overline{G}$. Hence $\gamma_g(G) \leq \gamma_{ed}(G)$.

Theorem: 2.20 If G is self-centered of diameter two, then $\gamma_g(G) \leq \gamma_{ed}(G)$.

Proof: Let D be an eccentric dominating set of G. If $v \in V-D$, there exist $u, w \in D$ such that $uv \in E(G)$ and w is eccentric to v. In $\overline{G}$, v and w are adjacent. So again, D is a global dominating set. Hence $\gamma_g \leq \gamma_{ed}$.

Remark: 2.3 Minimum eccentric dominating set need not be a minimum global dominating set.

In $\overline{K_2}+K_1+K_1+\overline{K_2}$, $\gamma_{ed} = 4$ and $\gamma_g = 3$..

Following result gives the exact bound for a product graph of a graph.

Theorem: 2.21 Let G be a connected graph with $|V(G)| = n$. Then $\gamma_{ed}(G \circ K_1) = n$.

Proof: Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let v_i' be the pendent vertex adjacent to v_i in $G \circ K_1$ for $i = 1, 2, \ldots, n$.

Then $\{v_1', v_2', \ldots, v_n'\}$ is an eccentric dominating set for $G \circ K_1$, and is also a minimum dominating set for $G \circ K_1$. Hence $\gamma_{ed}(G \circ K_1) = n$.

Next, we characterize some special classes of graph for which eccentric domination number is 1 or 2.

Theorem: 2.22 $\gamma_{ed}(G) = 1$ if and only if $G = K_n$.

Proof: If $G \neq K_n$, then G has atleast one pair of non-adjacent vertices with eccentricity greater than one.

Theorem: 2.23 Let G be a connected graph. Then $\gamma_{ed}(G) = 2$ if and only if G is any one of the following.

(i) $r(G) = 1$, $d(G) = 2$ and $u \in V(G)$ such that $e(u) = 2$ and $d(u, v) = 2$ for all $v \in V(G)$ with $e(v) = 2$.

(ii) G is self-centered of diameter 2, having a dominating edge which is not in a triangle.

(iii) $r(G) = 2$, $d(G) = 3$ and G has a γ-set D of cardinality two which is not connected.

Proof: When G satisfies any one of the above conditions obviously $\gamma_{ed}(G) = 2$.

On the other hand, assume that $\gamma_{ed}(G) = 2$. Therefore, $\gamma(G) = 1$ or $\gamma(G) = 2$.

Case(i) $\gamma(G) = 1$ and $\gamma_{ed}(G) = 2$. This implies G satisfies (i).

Case(ii) $\gamma(G) = 2 = \gamma_{ed}(G)$

Let D be a minimum γ_{ed}-dominating set of G. Let $D = \{u, v\} \subseteq V(G)$.

Since $\gamma(G) = 2$, $r(G) \geq 2$.

(a) <D> is connected:

Since D is connected u and v are adjacent and the edge uv is a dominating edge for G. Therefore $r(G) \geq 2$ and $2 \leq d(G) \leq 3$. Suppose $d(G) = 3$, there exists a vertex x with eccentricity 3 and x is dominated by u or v.

Let $xu \in E(G)$. Now, D is an γ_{ed}-set. Hence v must be an eccentric point of x. This implies that $d(x, v) = 3$, But x u v is a path $\Rightarrow d(x, v) = 2$, which is a contradiction. Hence, x must be a vertex with eccentricity 2. This implies that $d(G) = 2$, that is G is self-centered with diameter 2. [There exists no w, adjacent to both u and v, since in that case, w has no eccentric point in D, since $r(G) \geq 2$]

(b) <D>is not connected:

G is a connected graph, $\gamma(G) = 2 = \gamma_{ed}(G)$ implies that $d(G) \leq 3$. Therefore $d(u, v) = 2$ or 3.

If $d(u, v) = 2$, there exists $w \in V(G)$ such that w is adjacent to both u and v. Therefore, w must be of eccentricity one, since $D = \{u, v\}$ is an eccentric dominating set. Thus G is a graph with $r(G) = 1$ and $d(G) = 2$ which is a contradiction to $\gamma(G) = 2$. Hence $d(u, v) \neq 2$. This implies that $d(u, v) = 3$. Then $e(u) = e(v) = 3$ and uwxv is a shortest path, since D is an eccentric dominating set, eccentric point of w must be v and eccentric point of x must be u. Therefore, $e(w) = e(x) = 2$. Thus G is a connected graph with radius 2 and diameter 3 with a γ-set D of cardinality two which is not connected. This proves the theorem.

Corollary 2.23: $\gamma(P_4) = 2$ and $\gamma(K_{1,n}) = 2$.

Theorem: 2.24 $\gamma_{ed}(G) = 2$ and D is an eccentric dominating set $\cong K_2$ if and only if every vertex in V–D is adjacent to exactly one of u and v and G is self-centered with diameter 2.

Proof: Follows from case (ii)(a) of previous theorem.

Theorem: 2.25 If G is a graph with diameter three and $\gamma(G) = 2$ then $\gamma_{ed}(G) = 2$ if and only if G has a pair {u, v} such that one is the unique eccentric point of other and d(u, v) = 3

Proof: Follows from case (ii)(b) of previous theorem.

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