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On The Chromatic Preserving Sets

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Abstract: Let G be a simple graph with vertex set V(G) and edge set E(G). A Set $S \subseteq V$ is said to be a chromatic preserving set or a cp-set if $\chi(\langle S\rangle) = \chi(G)$ and the minimum cardinality of a cp-set in G is called the chromatic preserving number or cp-number of G and is denoted by cpn(G). A cp-set of cardinality cpn(G) is called a cpn-set. A partition of V(G) is said to be a cp-partition, if each subset in the partition induces a chromatic preserving set (cp-set). The cp-partition number of a graph G is defined to be the maximum cardinality of a cp-partition of V(G) and is denoted by cppn(G). In this paper, cp-number and cp-partition number of some standard graphs are found. Some of the graphs for which cpn(G) = $\chi(G)$ are identified. Some Nordhaus-Gaddum type of results are obtained for cp-number and cp-partition number.

Keywords: Chromatic preserving set, chromatic preserving number, cp-partition, cp-partition number
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1. Introduction

Graphs considered in this paper are finite, simple and undirected. For any graph G, the vertex set and edge set are denoted by V(G) and E(G) respectively.

A *clique* of a graph G is a maximal complete sub graph. The cardinality of a maximum clique is called the ***clique number*** and is denoted by $\omega(G)$. A ***wheel*** W_n is obtained by joining each vertex of C_{n-1} to an isolated vertex. If S is a non empty subset of the vertex set of a graph G, then the sub graph of G induced by S is the graph with vertex set S and edge set consisting of all those edges of G with both the end vertices in S and is denoted by <S>. A ***block*** of a graph G is a maximal, 2-connected sub graph of G. A graph G is a ***block graph*** if and only every block of G is a complete graph. A graph G is said to be a ***perfect graph*** if $\chi(H) = \omega(H)$ for all induced sub graphs H of G . A graph G is ***chordal*** or ***triangulated*** if every cycle of length greater than three has a chord. A set of vertices in a graph G is ***independent*** if no two of them are adjacent in G. The maximum cardinality among such independent sets is called the ***independence number*** of G and is and denoted by $\beta_o(G)$. An independent set of edges of G has no two of its edges adjacent and the maximum cardinality of such a set is the ***edge independence number*** $\beta_1(G)$.

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A matching in a graph is a set of independent edges and a perfect matching is a set of independent edges such that each vertex is an end vertex of some edge.

A set $S \subseteq V$ is a ***dominating set*** of G if for each $u \in V - S$, there exists a vertex $v \in S$ such that u is adjacent to v. The minimum cardinality of a dominating set in G is called the ***domination number*** and is denoted by $\gamma(G)$. A dominating set $S \subseteq V$ of G is a ***total dominating set*** if <S> has no isolated vertices. The minimum cardinality of a total dominating set in G is called the ***total domination number*** and is denoted by $\gamma_t(G)$. A dominating set $S \subseteq V$ of G is a ***connected dominating set*** if <S> is a connected subgraph of G. The minimum cardinality of a connected dominating set in G is called the ***connected domination number*** and is denoted by $\gamma_c(G)$.

A ***k-coloring*** of a graph G is a labeling $f : V \longrightarrow \{1,2,\ldots,k\}$. The labels are colors; the vertices with color i form a color class. A k-coloring is ***proper*** if $xy \in E$ implies $f(x) \neq f(y)$. A graph G is ***k-colorable*** if it has a proper k-coloring. The ***chromatic number*** $\chi(G)$ is the minimum k such that G is k-colorable. If $\chi(G) = k$, then G is said to be ***k-chromatic***. If $\chi(G) = k$, but $\chi(G) < k$ for every proper sub graph H of G, then G is said to be a ***k-color-critical graph***. A graph G is said to be a ***vertex-color-critical graph*** or ***k-critical graph*** if $\chi(G - u) < \chi(G)$ for every $u \in V$. Critical graphs were defined by Dirac [1]. In the literature, there are many questions posed by mathematicians on critical graphs. The book by Jensen and Toft [4] lists all famous problems on critical graphs. The k-critical graphs for k = 1, 2 and 3 are K_1, K_2 and odd cycles, respectively. For $k \geq 4$, the k-critical graphs have not been characterized. Ordinarily, it is extremely difficult to determine whether a given graph is critical; however every k-chromatic graph $k \geq 2$ contains a k-critical sub graph. In fact, if H is any smallest (in terms of number of vertices) induced sub graph of G such that $\chi(G) = \chi(H)$, then H is critical. Also not much is known on how to find the smallest critical sub graph of a non critical graph. Hermann [3] made an attempt to propose new exact algorithms for finding the chromatic number of a graph G. The algorithm attempts to determine the smallest possible induced sub graph H of G, which has same chromatic number as G. As mathematicians are more interested in rigorous proof techniques, in this paper we made an attempt to find the smallest subset of a vertex set, which induces a critical graph having the same chromatic number of the given graph. We define a vertex subset satisfying the above condition as a ***chromatic preserving set*** (or cp-set) and the minimum chromatic preserving set as ***cpn-set***. In a real life situation, finding a cpn-set gives a feasible solution to number of problems. For an example, in an Organization/Institution, the management is interested in forming a team from employees/students to train them in different specified skills so that (i) the members are reachable from each other (by reachable we mean that no sub team is isolated from the main team) (ii) there is at least one member for each specified skill (iii) one member is selected only for one skill (iv) two members with direct association are not selected for the same training (v) the minimum number of different skills that can be given to the group and (vi) the size of the team is as small as possible. Now, if a graph is drawn with the vertices representing the employees or students and an edge is drawn if an association exists between any two members. If different skills are marked by different colors, then the proper coloring of the graph satisfies (iii) and (iv). Finding the chromatic number of the graph satisfies condition (v). Finally, a minimal cp-set satisfies conditions (i) and (ii), and a cpn-set satisfies condition (vi).

Unless otherwise mentioned any graph considered in this paper is a (p, q)-graph. Definitions not given may be referred to [2] and [5].

2. Prior Results

Theorem 2.1 ([5], pp 177). If G is a k-color-critical graph, then $\delta(G) \geq k - 1$.

Theorem 2.2 ([2], pp 129). For any graph G, $\chi(G) + \chi(\overline{G}) \leq p + 1$.

Proposition 2.3 Graphs G and $\overline{G}$ are bipartite if and only if $G = C_4$, P_3, P_4, or $2K_2$.

Proof: If G has a vertex u such that $d(u) > 2$, then in $\overline{G}$, the neighbors of u form a 3-cycle. If $diam(G) > 3$, then in $\overline{G}$, a 3-cycle is induced. In both cases contradiction arises.

3. Main Results

3.1. Chromatic preserving sets in graphs

Definition 3.1.1. A set $S \subseteq V$ is said to be a chromatic preserving set or a cp-set if $\chi(\langle S\rangle) = \chi(G)$ and the minimum cardinality of a cp-set in G is called the chromatic preserving number or cp-number of G and is denoted by cpn(G). A cp-set of cardinality cpn(G) called cpn-set.

Example 3.1.2. The graph G given in Figure 1(Appendix II) is 3-chromatic graph with 10 vertices and 15 edges and it does not have a 3-cycle. From that Figure, it is clear that the minimum odd cycle is of length 5 and hence, cpn(G) = 5.

Observation 3.1.3. Properties of a minimal cp-set:

i. If $\chi(G) \geq 3$, then cpn-set induces a 2-connected vertex-color-critical graph.

ii. If G is connected, then cpn(G) = p if and only if G is vertex-color-critical.

iii. There does not exist any disconnected graph G such that cpn(G) = p.

iv. cpn(G) = 1 if and only if $G = nK_1$, $n \geq 1$.

v. G is a bipartite graph if and only if cpn(G) = 2.

vi. If G is 3-chromatic, then $cpn(G) = g_o(G)$.

vii. If cpn(G) = 3, then G is 3-chromatic.

viii. For any non-trivial graph G, which is neither vertex-color-critical nor totally disconnected, $2 \leq cpn(G) \leq p - 1$.

ix. For a disconnected graph G with k < p components, $cpn(G) \leq p - k + 1$.

x. If $\chi(G - u) < \chi(G)$ for a vertex $u \in V$, then u is in every minimal cp-set of G and conversely. Similarly, if $\chi(G - e) < \chi(G)$ for an edge $e \in E$, then e is in every minimal cp-set of G and conversely.

xi. If $\chi(G - u) = \chi(G)$, then $cpn(G - u) \geq cpn(G)$.

xii. If $cpn(G) \leq 4$, then G is a perfect graph.

The following Proposition gives the cp-number of some standard graphs.

Proposition 3.1.4.

i. $cpn(K_n) = n$;

ii. $cpn(C_n) = \begin{cases} 2, & \text{if n is even} \\ n, & \text{if n is odd;} \end{cases}$

iii. $cpn(W_n) = \begin{cases} 3, & \text{if n is even} \\ n, & \text{if n is odd;} \end{cases}$

Proposition 3.1.5. For any graph G,

i. $\omega(G) \leq \chi(G) \leq cpn(G)$.

ii. $\chi(G) = cpn(G)$ if and only if cpn-set induces a complete graph.

Proof:

i) Follows trivially.

ii) Suppose a cpn-set induces a complete graph and S is a cpn-set of G. Then $\langle S \rangle = K_r$ for some r and hence, cpn(G) = r. Further $\chi(G) = \chi(\langle S \rangle) = r$ and the result follows. Conversely, suppose $cpn(G) = \chi(G)$ and S is a cpn-set of G. Therefore, $|S| = cpn(G) = \chi(\langle G \rangle) = \chi(\langle S \rangle)$. Hence, S induces a complete graph.

Proposition 3.1.6. If G is a perfect graph, then $cpn(G) = \chi(G) = \omega(G)$.

Proof: If G is perfect, then $\omega(G) = \chi(G)$. Then a cpn-set induces a complete graph and the result follows from Proposition 3.1.5(ii).

Proposition 3.1.7. If G is a disconnected graph with components $G_1, G_2, \ldots, G_k$, then cpn(G) $= \min_i \{cpn(G_i) \mid \chi(G_i) = \chi(G)\}$.

Proposition 3.1.8. If S is a cpn-set of a graph G and $\chi(G) = k \geq 3$, then $\delta(\langle S \rangle) \geq k - 1$.

Proof: Since <S> is vertex-color-critical, from Theorem 2.1, the result follows.

Proposition 3.1.9. If a connected graph G has a dominating cpn-set and $\gamma(G) \geq 2$, then $\gamma(G) \leq \gamma_t(G) \leq \gamma_c(G) \leq cpn(G)$.

Proposition 3.1.10. A graph G is 3-chromatic with $g_o(G) = 5$ or a 5-chromatic graph containing K_5 as a maximal complete graph G if and only if cpn(G) = 5.

Proof: Necessary part is trivial. Now suppose cpn(G) = 5 and G is a k-chromatic graph. Clearly, $3 \leq k \leq 5$.

<u>Claim</u> : $k \neq 4$.

Suppose k = 4. Let S = {a, b, c, d, e} be a cpn-set of G. Since <S> is 4 -chromatic, exactly two of the vertices are of same color and the remaining three vertices are of different colors. Then one of the vertices of same color must be adjacent to all the remaining 3 colors. This adjacency induces a K_4, which consequently implies cpn(G) = 4, a contradiction.

<u>Case i</u>: k = 3.

From Observation 3.1.3(vi), cpn(G) = $g_o(G)$ and the result follows.

<u>Case ii:</u> k = 5.

Then cpn(G) = $\chi(G)$ and by Proposition 3.1.5(ii), cpn-set induces a complete graph and hence, G contains K_5 as a maximal complete sub graph.

Proposition 3.1.11. For a graph G,

i. cpn(G) = cpn $(\overline{G})$ = 1 if and only if G = K_1.

ii. cpn(G) = cpn $(\overline{G})$ = 2 if and only if G = P_3, $K_2 \cup K_1$, P_4, C_4, $2K_2$.

Proposition 3.1.12. If cpn(G) = cpn$(\overline{G})$ = 3, then $\chi(G) = 3$ and $p \geq 5$.

Proof: If cpn(G) = 3, then $\chi(G) = 3$ and G has a 3-cycle say u-v-w-u. By similar argument, $\overline{G}$ has a 3-cycle say x-y-z-x. Then x, y, z are independent vertices in G. Hence, at least 2 vertices of x, y, z cannot be in the 3-cycle u-v-w-u. Thus $p \geq 5$.

Theorem 3.1.13. For a connected graph G, cpn(G) = cpn($\overline{G}$) = 3 if and only if G and $\overline{G}$ are 3-chromatic and contain at least one of the sub graphs in the family of graphs given in Figure 2 (Appendix II) as induced sub graph.

Proof: Necessary part is trivial. Now suppose cpn(G) = cpn($\overline{G}$) = 3. Then G and $\overline{G}$ are 3-chromatic and hence, both G and $\overline{G}$ contain 3-cycles. Let a-b-c-a and d-e-f-d be 3-cycles in G and $\overline{G}$ respectively. Clearly, {d, e, f} forms an independent set in G and {a, b, c} forms an independent set in $\overline{G}$. Two cases arise. Here, the case is being discussed for the graph G. Similar discussion can be had for $\overline{G}$ also. Let ß be the family of graphs given in Figure 2 (Appendix II).

__Case i:__ {a, b, c} $\cap$ {d, e, f} $\neq \phi$.

Clearly, |{a, b, c} $\cap$ {d, e, f}| = 1. Without loss of generality, let a = f. Then {a, d, e} forms an independent set in G. Let S = {a, b, c, d, e}. Then $deg_{<S>}(a) = 2$, $2 \leq deg_{<S>}(b) \leq 4$ and $2 \leq deg_{<S>}(c) \leq 4$. Table 1 (a) (Appendix I) lists the graphs <S> for the various values of deg(b) and deg(c) with deg(a) = 2 in <S>.

__Case ii:__ {a, b, c} $\cap$ {d, e, f} = ϕ.

Let H = <{a, b, c, d, e, f}>. In H, the following facts are observed.

__Fact 1:__ At most two vertices of {a, b, c} can be adjacent to the same vertex of {d, e, f}.

Otherwise K_4 is induced.

__Fact 2:__ Each vertex of {a, b, c} is adjacent to a vertex of {d, e, f}.

Otherwise $\beta_o(H) = 4$. Then K_4 is induced in $\overline{G}$, a contradiction to $\overline{G}$ is a 3-chromatic graph.

__Fact 3:__ A vertex of {d, e, f} can be adjacent to at most two vertices of {a, b, c}.

Otherwise K_4 is induced.

__Sub case i:__ H is disconnected.

__Claim:__ H has only one isolated vertex.

Suppose H = $K_3 \cup 3K_1$. Then similar argument as in Fact 2 leads to a contradiction. So suppose H has 2 isolated vertices say d and e. From Fact 2 each vertex of {a, b, c} is adjacent to vertex f and inducing K_4, a contradiction. Hence, the claim holds.

Let d be the isolated vertex of H. then from Fact 2, 3 ≤ deg(a), deg(b), deg(c) ≤ 4 and from Fact 3, 1 ≤ deg(e), deg(f) ≤ 2. Then the following fact is observed.

__Fact 4:__ At most one vertex of {a, b, c} can be of degree 4.

Suppose deg(a) = deg(b) = 4. Then as deg(c) ≥ 3, a, b, c are adjacent to the same vertex of {d, e, f}, contradicting Fact 1.

If deg(a) = deg(b) = deg(c) = 3, then graph(h) is induced. Suppose deg(a) = deg(b) = 3, deg(c) = 4. From Fact 1, graph (i) is induced.

Sub case ii: H is connected.

From Fact 3, 1 ≤deg(d), deg(e), deg(f) ≤ 2 and from Fact 2, 3 ≤deg(a), deg(b), deg(c)≤ 5. Then the following facts are observed.

Fact 5: At most one vertex of {a, b, c}can be of degree 5.

Suppose deg(a) = deg(b) = 5. Then from Fact 1, deg(c) = 2 contradicting deg(c) ≥ 3.

Fact 6: Two vertices of {a, b, c} are of degree 4 and one vertex of {a, b, c} is of degree 5 cannot hold. Suppose fact 6 does not hold. Then sum of the degrees of vertices d, e and f is 7, a contradiction to the fact that sum of the degrees is at most 6 as 1 ≤ deg(d), deg(e), deg(f) ≤ 2.

Table 1 (b) (Appendix I) lists the graph H for the various values of deg(a), deg(b) and deg(c).

Proposition 3.1.14. If for a graph G, cpn(G) = cpn($\overline{G}$) = 3, then β_o(G) = β_o($\overline{G}$) = 3.

Proof: cpn($\overline{G}$) = 3 implies that β_o(G) $\geq$ 3. If β_o(G) $\geq$ 4, then $\overline{G}$ contains K_4 as an induced sub graph, a contradiction. Hence, β_o(G) = 3. Similarly, β_o($\overline{G}$) = 3.

Definition 3.1.15. A graph G is called a well colored graph if all minimal cp-sets have the same cardinality.

Observation 3.1.16.

i. Totally disconnected graph is a well colored graph.

ii. Bipartite graph is a well colored graph.

iii. Triangulated graph is a well colored graph.

iv. Block graph is a well colored graph.

Proposition 3.1.17. If G is a block graph with a single cut vertex, then $\overline{G}$ is a well colored graph.

Proof: Let u be the cut vertex of G and G - u = $G_1 \cup G_2 \cup \ldots \cup G_k$. Let $|V(G_i)| = n_i$, 1 ≤ i ≤ k. Then $\overline{G} = K_{n_1,n_2,\ldots,n_k}$ and the result follows.

3.2. Cp-partition of graphs

Two new parameters cp-partition and cp-partition number are defined in this section.

Definition 3.2.1. A partition of V(G) is said to be a cp-partition, if each subset in the partition induces a chromatic preserving set (cp-set). The cp-partition number, cppn(G) is defined to be the maximum cardinality of a cp-partition of V(G).

Proposition 3.2.2.

i. $cppn(K_n) = 1$;

ii. $cppn(nK_1) = n$;

iii. $cppn(K_{1,n}) = 1$;

iv. $cppn(K_{m,n}) = \min\{m, n\}$, $m, n \geq 2$;

v. $cppn(C_n) = \begin{cases} n/2, & \text{if n is even} \\ 1, & \text{if n is odd;} \end{cases}$

vi. $cppn(W_n) = 1$.

Observation 3.2.3.

i. If G is a vertex-color-critical graph, then cppn(G) = 1. The converse need not be true.

ii. For each pair of positive integers n and r, there exists a graph G with cpn(G) = n and cppn(G) = r.

Proof:

i) Consider the graph $G = K_n - e$ for any n. Then cpn(G) = n - 1, and hence, cppn(G) = 1. Clearly, G is not vertex-color-critical.

ii) A complete n-partite graph $K_{r,r,\ldots,r}$ satisfies the required properties.

Proposition 3.2.4.

i. If G is a bipartite graph, then $cppn(G) = \beta_1(G)$.

ii. If G is a connected graph with $\chi(G) = k$, then $cppn(G) \leq \frac{p}{k}$.

iii. If G is not totally disconnected, then $cppn(G) \leq \frac{p}{2}$.

iv. cppn(G) = p if and only if $G = pK_1$.

v. If G is bipartite, then $cppn(G) \leq \frac{p}{2}$ and the equality holds if and only if G has a perfect matching.

Proof: Trivial.

Proposition 3.2.5. For a connected graph G,

i. cpn(G) = cppn(G) = 1 if and only if $G = K_1$.

ii. cpn(G) = cppn(G) = 2 if and if and only if $G = P_5$, P_4, C_4 or G∈ ß, where ß is the family of graphs given in Figure 3 (Appendix II).

3.3. Some Nordhaus-Gaddum type of results

Proposition 3.3.1. For a perfect graph G, $cpn(G) + cpn(\overline{G}) \leq p + 1$.

Proof: G is perfect if and only if $\overline{G}$ is perfect. From Proposition 3.1.6, cpn(G) = χ(G) and cpn($\overline{G}$) = $\chi(\overline{G})$. Hence, cpn(G) + cpn($\overline{G}$) = χ(G) + $\chi(\overline{G}) \leq$ p + 1.

Proposition 3.3.2. If G is vertex-color-critical, then cpn(G) + cpn($\overline{G}$) = p + 1 if and only if G is complete.

Proof: If G is complete, G = K_p and hence, $\overline{G}$ = pK_1. Hence, χ(G)= cpn(G)=p; $\chi(\overline{G})$=cpn($\overline{G}$)=1 and the result follows. Suppose cpn(G) + cpn($\overline{G}$) = p +1. Since G is vertex-color-critical, cpn(G) = p. Hence, cpn($\overline{G}$) = 1. Thus, $\overline{G}$ = pK_1.Therefore, G = K_p.

Proposition 3.3.3. If G is a perfect, then cpn(G) + cpn($\overline{G}$) $\leq$ p + 1.

Proof: If G is perfect, then $\overline{G}$ is perfect. From Proposition 3.1.6., cpn(G) = χ(G) and cpn($\overline{G}$) = $\chi(\overline{G})$. Then the result follows from Theorem 2.2.

Proposition 3.3.4. If G is bipartite and $\overline{G}$ is disconnected, then cpn(G) + cpn($\overline{G}$) = p + 1 if and only if G = $K_{1,p-1}$, p $\geq$ 3.

Proof: Necessary condition is trivial. Suppose cpn(G) + cpn($\overline{G}$) = p + 1. Since cpn(G) = 2, cpn($\overline{G}$) = p - 1. As $\overline{G}$ is disconnected, one component H of $\overline{G}$ has at least p - 1 vertices and $\chi(\overline{H})$ = $\chi(\overline{G})$. Since $\overline{G}$ has p vertices, and at least two components, $\overline{H}$ has exactly p - 1 vertices. Thus, $\overline{G}$ = $\overline{H} \cup K_1$. Consequently G = H + K_1. Since G is bipartite and K_1 is joined to all vertices of H, H must be a null graph. Hence, G = $K_{1,\ p-1}$.

Proposition 3.3.5. If G and $\overline{G}$ are 3-chromatic, then cpn(G) + cpn($\overline{G}$) = 2p if and only if G = C_5.

Proof: Necessary part is trivial. Suppose cpn(G) + cpn($\overline{G}$) = 2p. Then cpn(G) = cpn($\overline{G}$) = p. Therefore, G and $\overline{G}$ are vertex-critical graphs and hence, they are odd cycles. Clearly G, $\overline{G}$ $\neq$ C_3. Suppose G $\neq$ C_5. Then G = C_{2r+1}, r $\geq$ 3. This implies β_o(G) $\geq$ 3 and hence, $g_o(\overline{G})$ = 3, a contradiction.

Proposition 3.3.6. If G is a vertex-color-critical graph, then cpn(G) + cpn($\overline{G}$) = p + 1 if and only if G is complete.

Proof: If G is complete, then $\overline{G}$ is totally disconnected graph and thus, the result follows. Suppose

$cpn(G) + cpn(\overline{G}) = p + 1$. Since G is vertex-color-critical, cpn(G) = p and therefore, $cpn(\overline{G}) = 1$. Thus, $\overline{G}$ is a 1-chromatic graph. Hence, $\overline{G}$ is either a trivial graph or a totally disconnected graph and hence, G is complete.

Observation 3.3.7. There exist k-chromatic graphs G and $\overline{G}$ such that $cpn(G) + cpn(\overline{G}) = 2k$.

Example 3.3.8. Let G be a graph obtained from K_k by adding pendant edges added at each vertex of K_k. Then both the graphs G and $\overline{G}$ are p-chromatic and contain K_k as an induced sub graph. Therefore, $cpn(G) = cpn(\overline{G}) = p$.

Proposition 3.3.9. If G and $\overline{G}$ are bipartite, then $cppn(G) + cppn(\overline{G}) = p$ if and only if G $= C_4, P_4$, or $2K_2$.

Proof: Necessary condition is trivial. Suppose $cppn(G) + cppn(\overline{G}) = p$. From Proposition 2.3, G and $\overline{G}$ are bipartite if and only if $G = C_4, P_3, P_4$, or $2K_2$. If $G = C_4, P_3, 2K_2$ or P_4, then $\overline{G} = 2K_2, K_2 \cup K_1, C_4$, and P_4 respectively. If $G = P_3$, then $cppn(G) = cppn(\overline{G}) = 1$, a contradiction. If $G = C_4, P_4$, or $2K_2$, then $cppn(G) = cppn(\overline{G}) = 2$, and the result follows.

Proposition 3.3.10. If G is a vertex-color-critical graph, then $cppn(G) + cppn(\overline{G}) = p + 1$ if and only if G is complete.

Proof: Let G be a complete graph. Then cpn(G) = p and $cpn(\overline{G}) = 1$.Therefore, cppn(G) = 1, $cppn(\overline{G}) = p$ and hence, $cppn(G) + cppn(\overline{G}) = p + 1$. Conversely suppose cppn(G) + $cppn(\overline{G}) = p+1$. Since G is vertex-color-critical graph, cpn(G) = p and hence, cppn(G) = 1 This implies $cppn(\overline{G}) = p$. Therefore, $\overline{G}$ is a totally disconnected graph and hence, G is complete.

Proposition 3.3.11. If G is neither complete nor totally disconnected, then cppn(G) + $cppn(\overline{G}) \leq p$.

Proof: Since G is not totally disconnected, $cppn(G) \leq \frac{p}{2}$. As G is not complete, $\overline{G}$ is not totally disconnected and hence, $cppn(\overline{G}) \leq \frac{p}{2}$. Hence, $cppn(G) + cppn(\overline{G}) \leq p$.

4. Problems

1. Is it possible to characterize all k-chromatic graphs G and $\overline{G}$ of order p such that $cpn(G) + cpn(\overline{G}) = 2p$.
2. For a k-chromatic graph G satisfying the condition in problem 1, what is the maximum number of edges that can be added to G so that the property is maintained.

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Appendix I

Table 1(a)

$\deg_{<S>}(b)$	$\deg_{<S>}(c)$	<S>
2	2	(a)
2	3	(b)
2	4	(c)
3	3	(d), (e)
3	4	(f)
4	4	(g)

Table 1(b)

$\deg_H(a)$	$\deg_H(b)$	$\deg_H(c)$	H
3	3	3	(j)
3	3	4	(k), (l)
3	3	5	(m)
3	4	4	(n), (o)
3	4	5	(p)
4	4	4	(q)

Appendix II

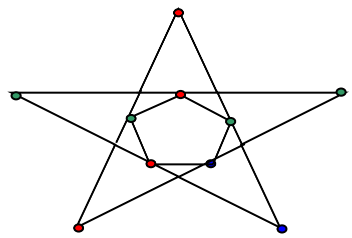
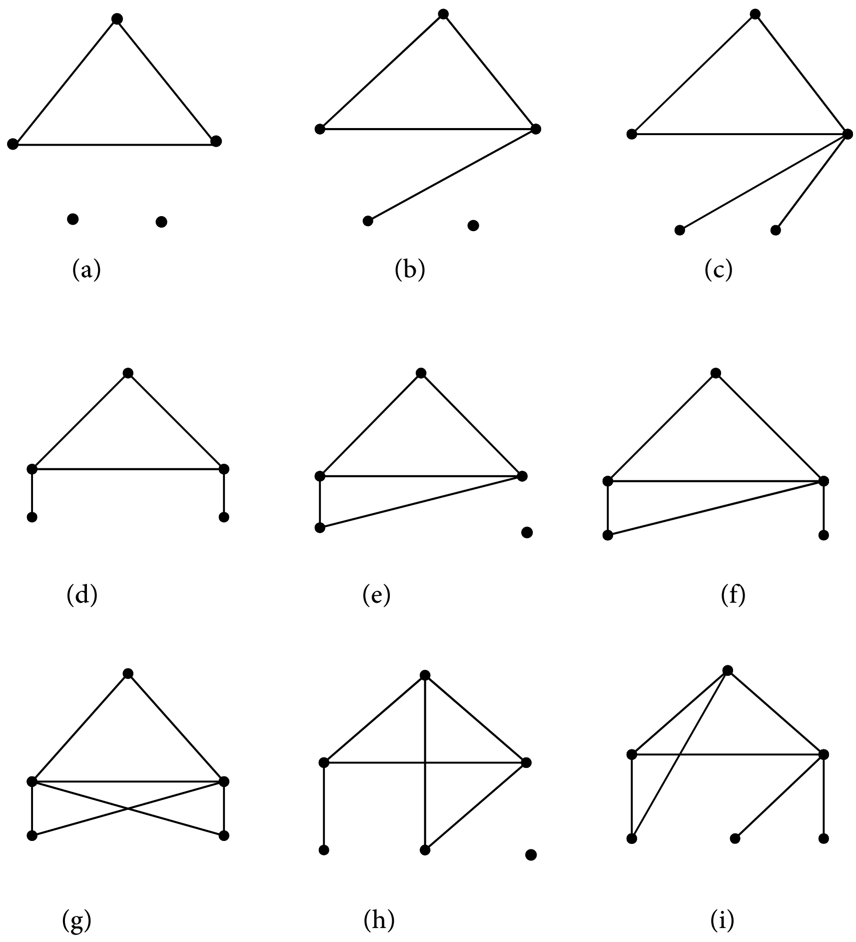


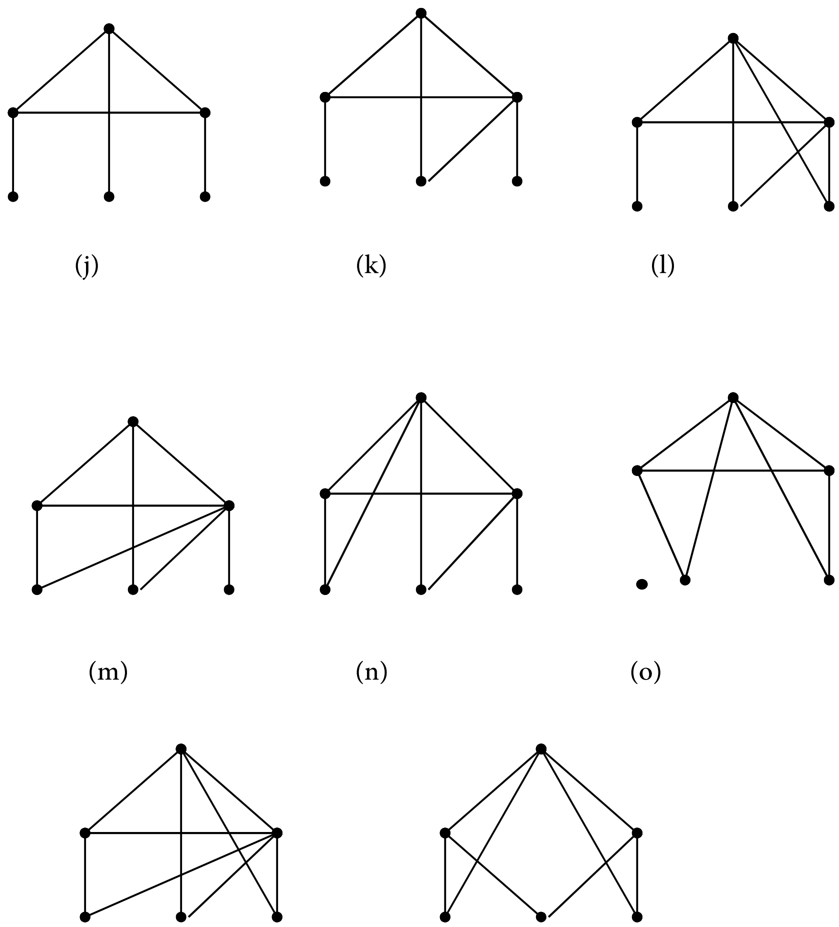
Figure 1



(a) (b) (c)

(d) (e) (f)

(g) (h) (i)

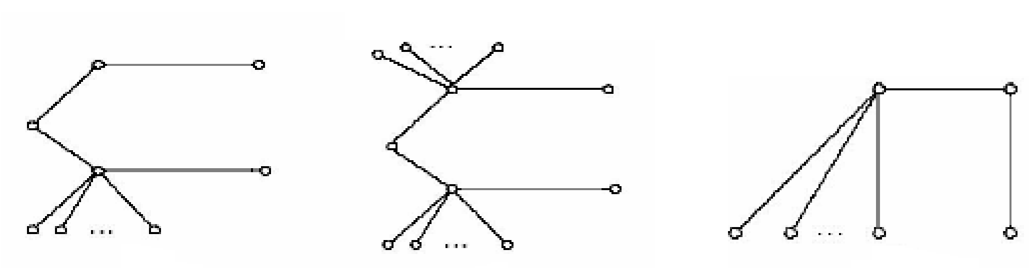


(j) (k) (l)

(m) (n) (o)

(p) (q)

Figure 2



(a) (b) (c)

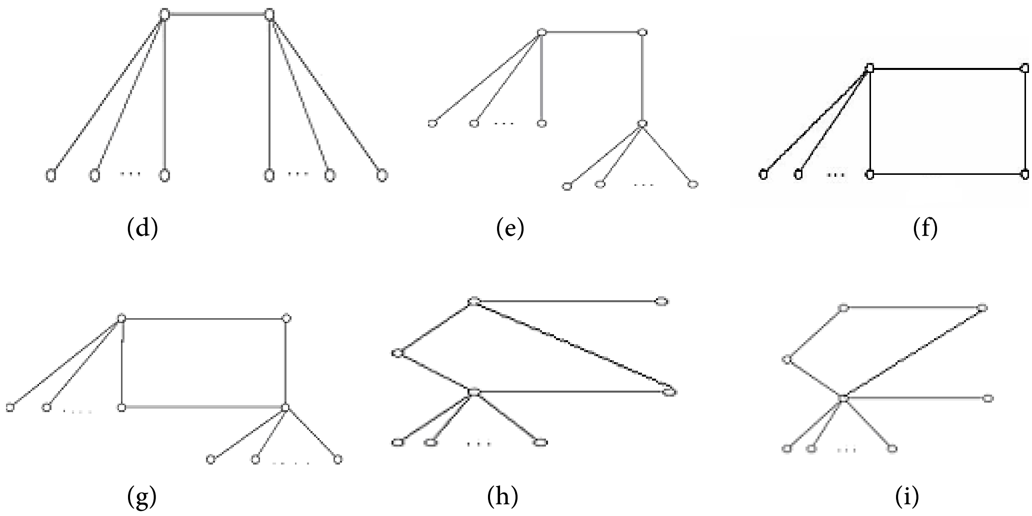


Figure 3