International Journal of Engineering Science, Advanced Computing and Bio-Technology
Vol. 1, No. 1, January –March 2010, pp. 19-28

Domination Parameters Of Hypercubes

T.N.Janakiraman[1], M.Bhanumathi[2] and S.Muthammai[2]

[1]*Department of Mathematics and Computer Applications,*

National Institute of Technology, Triuchirapalli, 620015,TamilNadu, India

E-Mail: *janaki@nitt.edu, tnjraman2000@yahoo.com*

[2]*Government Arts College for Women, Pudukkottai.622001, TamilNadu, India.*

E-Mail: *bhanu_ksp@yahoo.com, muthammai_s@yahoo.com*

Abstract: *Let G be a connected graph. Let γ, γ_i, γ_t, γ_c and γ_p denote respectively the domination number, the independent domination number, the total domination number, the connected domination number and the perfect domination number of G. The n-cube Q_n is the graph, whose vertex set is the set of all n-dimensional Boolean vectors with two vertices being joined if and only if they differ in exactly one coordinate. Hamming proved that Q_n has a perfect dominating set if and only if $n = 2^k-1$. Here, it is proved that $\gamma(Q_n) = 2^{n-k}$ for $n = 2^k$. Bounds for $\gamma_i(Q_n)$, $\gamma_c(Q_n)$, $\gamma_t(Q_n)$ and $\gamma_p(Q_n)$ are also found out. Finally, it is conjectured that $\gamma(Q_n) = \gamma(Q_{n-1}) + \lceil (2^{n-1}-\gamma(Q_{n-1}))/(n-1) \rceil$ for $2^k+1 \le n \le 2^{k+1}-2$, where $\lceil x \rceil$ denote the least integer not less than x.*

1. Introduction

Let G be a finite, connected, undirected, simple graph with vertex set V(G) and edge set E(G). A vertex u is said to dominate the vertex v if E(G) contains an edge from u to v or if u = v. A set D $\subseteq$ V(G) is a *dominating set*, if every vertex in V(G) is either an element of D or is adjacent to an element of D; that is every vertex of G is dominated by at least one member of D. A dominating set D is an *independent dominating* set, if no two vertices in <D> are adjacent, that is, D is an independent set. A dominating set D is a *connected dominating* set, if < D > is a connected subgraph of G. A dominating set D is a *perfect dominating* set, if each vertex of G is dominated by exactly one element of D. Clearly every perfect dominating set is independent dominating set. A dominating set D is a *total dominating* set, if < D > has no isolated vertex. The domination number γ of G is defined to be the minimum cardinality of a dominating set in G. Similarly, we can define the perfect domination number γ_p, connected domination number γ_c, total domination number γ_t, independent domination number γ_i for a graph G. It is clear that a perfect dominating set for a graph is necessarily a minimum dominating set. A dominating set with cardinality γ(G) is known as a γ*-dominating set*. The *domatic number* d(G) of a graph G is the maximum number of elements in a partition of V(G) into dominating sets. The *distance* d(u, v) between two vertices u and v in G is the minimum length of a path joining them. Let D_1 and D_2 be two subsets of V(G). Distance between the two sets D_1, D_2

Received: 08 February, 2009; Revised: 23 February, 2009; Accepted: 3 March, 2009

is defined as the minimum of $\{d(u, v) : u \in D_1, v \in D_2\}$. The definitions and details not furnished here may be found in [2] and [5]. The hypercube or n-cube Q_n is the graph whose vertex set is the set of all n-dimensional Boolean vectors in which two vertices are joined if and only if they differ in exactly one coordinate. We observe that $Q_1 = K_2$ and $Q_n = Q_{n-1} \times K_2$ if $n \geq 2$.

A communication network can be represented by a connected graph G, where the vertices of G represent processors and edges represent bi-directional communication channels. The hypercube Q_n of dimension n is one of the most versatile and powerful interconnection networks. It has been successfully employed in the architecture of massively parallel computers. A dominating set in Q_n can be interpreted as a set of processors from which information can be passed on to all the other processors. Hence, the determination of the domination parameters of Q_n is a significant problem.

The notion of perfect dominating set in a hypercube is same as that of a single-error correcting binary code, which is due to R.W.Hamming[4]. Jha [6] has proved the following theorem:

Theorem 1.1[6] Let $\gamma(Q_n)$ denote the domination number of Q_n. Then $[2^n/(n+1)] \leq \gamma(Q_n) \leq [2^n/2^h]$, where $h = \lfloor \log_2(n+1) \rfloor$. If $n = 2^k - 1$, then the two bounds coincide and hence $\gamma(Q_n) = 2^{n-k}$. These bounds are correct also for $\gamma_i(Q_n)$.

R.W.Hamming [4] proved the following theorem,

Theorem 1.2 [4] Q_n has a perfect single-error correcting code if and only if $n = 2^k - 1$. Arumugam et al [1] have independently proved the same theorem.

Zelinka [8] has proved the following theorem:

Theorem 1.3[8] Let k be a positive integer. Then the graph of the cube of dimension $2^k - 1$ and the graph of the cube of dimension 2^k have both the domatic number 2^k.

Jaeun Lee has proved the following:

Theorem 1.4[7] Let n be a natural number. Then the following are equivalent. (1) The hypercube Q_n has an independent perfect dominating set. (2) $n = 2^m - 1$ for a natural number m.

Here, we independently prove that $\gamma(Q_n) = 2^{n-k}$ when $n = 2^k$, and we obtain bounds for γ, γ_i, γ_t, and γ_c whenever $2^k \leq n \leq 2^{k+1} - 2$. We also conjecture that, $\gamma(Q_n) =$

$\gamma(Q_{n-1}) + \lceil (2^{n-1}-\gamma(Q_{n-1}))/(n-1) \rceil$ for $2^k+1 \leq n \leq 2^{k+1}-2$, where $\lceil x \rceil$ denote the least integer not less than x.

For brevity, take $m = 2^k$ throughout this paper. Thus $2m = 2^{k+1}$

2.Structure of the Hypercubes Q_n

The n-cube Q_n is the graph whose vertex set is the set of all n-dimensional Boolean vectors; two vectors being joined if and only if they differ in exactly one coordinate. We observe that $Q_1 = K_2$, $Q_n = Q_{n-1} \times K_2$, if $n \geq 2$. Now, Q_n can be viewed as follows:

Let $t = 2^{n-1}$ and let $(Q_{n-1})^0$, $(Q_{n-1})^1$ be two copies of Q_{n-1} with vertex sets $V((Q_{n-1})^0) = \{u_1^0, u_2^0,\ldots, u_t^0\}$, $V((Q_{n-1})^1) = \{u_1^1, u_2^1,\ldots, u_t^1\}$, where $u_i \in Q_{n-1}$.

Then $V(Q_n) = V((Q_{n-1})^0) \cup V((Q_{n-1})^1)$,

$E(Q_n) = E((Q_{n-1})^0) \cup E((Q_{n-1})^1) \cup \{ u_i^0u_i^1 : i = 1, 2,\ldots, t\}$. Similarly,

$Q_{n+1} = Q_n \times K_2 = (Q_{n-1} \times K_2) \times K_2$. Let $(Q_{n-1})^{00}$, $(Q_{n-1})^{10}$, $(Q_{n-1})^{01}$, $(Q_{n-1})^{11}$ be 2^2 copies of Q_{n-1} with $V((Q_{n-1})^{00}) = \{ u_1^{00}, u_2^{00},\ldots, u_t^{00} \}$, $V((Q_{n-1})^{10}) = \{ u_1^{10}, u_2^{10},\ldots, u_t^{10} \}$, $V((Q_{n-1})^{01}) = \{ u_1^{01}, u_2^{01},\ldots, u_t^{01} \}$, $V((Q_{n-1})^{11}) = \{ u_1^{11}, u_2^{11},\ldots, u_t^{11} \}$, and

$$V(Q_{n+1}) = V((Q_{n-1})^{00}) \cup V((Q_{n-1})^{01}) \cup V((Q_{n-1})^{10}) \cup V((Q_{n-1})^{11}),$$

$$E(Q_{n+1}) = E((Q_{n-1})^{00}) \cup E((Q_{n-1})^{10}) \cup E((Q_{n-1})^{01}) \cup E((Q_{n-1})^{11})$$
$$\cup \{ u_i^{00}u_i^{01} : u_i \in Q_{n-1}\} \cup \{ u_i^{00}u_i^{10} : u_i \in Q_{n-1}\}$$
$$\cup \{ u_i^{10}u_i^{11} : u_i \in Q_{n-1}\} \cup \{ u_i^{01}u_i^{11} : u_i \in Q_{n-1}\}.$$

That is, edges of Q_{n+1}'s are just the union of edges of $(Q_{n-1})^{00}$, $(Q_{n-1})^{10}$, $(Q_{n-1})^{01}$, $(Q_{n-1})^{11}$ and the edges joining u_i^{xy} and u_i^{pq}, where (x,y) , (p,q) are two dimensional Boolean vectors differing exactly in one place. Let $(Q_{n-1})^{000}$, $(Q_{n-1})^{100}$, $(Q_{n-1})^{010}$, $(Q_{n-1})^{110}$, $(Q_{n-1})^{001}$, $(Q_{n-1})^{101}$, $(Q_{n-1})^{011}$, $(Q_{n-1})^{111}$ be 2^3 copies of Q_{n-1} and $V((Q_{n-1})^{xyz}) = \{u_1^{xyz}, u_2^{xyz}, \ldots, u_t^{xyz}\}$. Then $V(Q_{n+2}) = \cup\ V((Q_{n-1})^{xyz})$, $E(Q_{n+2}) = \cup\ E((Q_{n-1})^{xyz}) \cup \{u_i^{xyz}u_i^{pqr}$:(x,y,z), (p,q,r) denote Boolean vectors differing at exactly one place$\}$. Similarly, we can view Q_{n+3} ,…etc, in terms of Q_{n-1} or Q_{n+3}, Q_{n+4}… etc in terms of Q_n etc.

Q_n consists of 2^k pairwise vertex-disjoint copies of Q_{n-k}.

For each binary k-tuple x in Q_k and for each binary (n-k)-tuple y in Q_{n-k}, let f(x, y) = xy, the concatenation of x and y. Clearly f is a 1-1 correspondance between the Cartesian product $Q_k \times Q_{n-k}$ and Q_n, and it is easy to see that f is in fact edge preserving and therefore a graph isomorphism. Define $(Q_{n-k})^x$ to be $\{xy \mid y \in Q_{n-k}\}$. The family $\{(Q_{n-k})^x \mid x \in Q_k\}$ gives the desired vertex partition of Q_n into 2^k copies of Q_{n-k}.

Proposition 2.1 Let S be any collection of k-dimensional Boolean vectors with $k \geq 4$ such that any two of them differ in exactly two places. Then S contains exactly k elements.

Proof: .Let x be any vertex of Q_k, and let S_1 be the set of all neighbors of x. Then $|S_1| = k$, and if $y,z \in S_1$, then d(y, z) = 2 (since y,x,z is a path). Therefore, S_1 is a collection of k-dimensional Boolean vectors such that any two of them differ in exactly two places. Hence $|S| \geq k$ —— (1)

Suppose S contains more than k elements, S contains at least k+1 elements x_1, x_2, …, x_{k+1}.

Claim: There exists at least one pair x_i, x_j, i, j >1 , $i \neq j$ such that x_i and x_j will not differ in exactly two places.

Proof of the claim:

Think of the vertices of Q_k as the subset of [k] = {1, 2, 3, …, k}. By the vertex symmetry of Q_k, we may assume that $x_{k+1} = 0$. Then for $1 \leq i \leq k$, x_i is a 2-subset of [k]. By assumption, x_i and x_j differ in exactly 2 places, i.e. $|x_i \Delta x_j| = 2$. Now $|x_i \Delta x_j| = |x_i| + |x_j| - 2|x_i \cap x_j| = 2 + 2 - 2|x_i \cap x_j|$. Thus $|x_i \cap x_j| = 1$. Without loss of generality, we may assume that $x_1 = \{1, 2\}$. We claim: either $1 \in x_i$ for all $2 \leq i \leq k$ or $2 \in x_j$ for all $2 \leq j \leq k$. For if not, there exists an i such that $x_i = \{2, a\}$ and there exists a j such that $x_j = \{1, b\}$, with a, $b \geq 3$. Since $|x_i \cap x_j| = 1$, a = b. So $x_j = \{1, 3\}$ and $x_i = \{2, 3\}$. Since $k \geq 4$, there exists an x_l with $l \neq 1, i, j$.

Case 1: $1 \in x_l$. Hence $2 \notin xl$ (or else xl = x1). Then since $|x_i \cap x_j| = 1$, $x_i \cap x_j = \{1\}$ and $3 \notin x_l$. Then $x_i \cap x_j = \phi$, a contradiction.

Case 2: $2 \in x_l$. Hence $1 \notin x_l$ Also, $3 \notin x_l$ since otherwise $x_l = x_i$. Then since $|x_i \cap x_l| = 1$, $x_l \cap x_j = \phi$, again a contradiction.

This proves our claim. Without loss of generality, assume $1 \in x_i$ for all i = 1, 2, …, k. But there are only k−1 2-subsets of [k] which contain 1 as a member. This contradiction proves that S cannot have k+1 elements. Hence, $|S| \leq k$.

From (1) and (2), we see that $|S| = k$..

Remark 2.1 We know $Q_n = Q_{n-1} \times K_2$. $V(Q_n) = V((Q_{n-1})^0) \cup V((Q_{n-1})^1)$ and $E(Q_n) = E((Q_{n-1})^0) \cup E((Q_{n-1})^1) \cup \{u_i^0 u_i^1 : i = 1, 2, 3, ..., 2^{n-1}, u_i^0 \in V((Q_{n-1})^0) , u_i^1 \in V((Q_{n-1})^1)\}$. Let D be a dominating set of Q_{n-1}. Let D^0, D^1 be the corresponding dominating sets of $(Q_{n-1})^0$, $(Q_{n-1})^1$. Then (i) $D^0 \cup D^1$ is a total dominating set for Q_n,

(ii) $D^0 \cup \{V((Q_{n-1})^1) - D^1\}$ is a dominating set for Q_n. Also if $S^1 \subseteq V((Q_{n-1})^1)$, then $D^0 \cup S^1$ is a dominating set for Q_n if and only if S^1 dominates $< V((Q_{n-1})^1) - D^1 >$.

Proposition 2.2 If $n = 2^k - 1$, vertices of Q_n can be partitioned into 2^k (= m) perfect dominating sets $D_1, D_2, D_3, ..., D_m$ each containing exactly 2^{m-k-1} elements.

Proof: For if D_1 is the perfect single-error correcting Hamming code on Q_n, D_1 is linear, i.e. a subgroup of Q_n under vector addition. We take $D_2, D_3, \ldots, D_m$ to be the other cosets of D_1 in Q_n, thereby giving a partition of Q_n into perfect dominating sets, all translates of D_1 under addition. Thus all are perfect dominating sets, and have the same size.

Remark 2.2 Consider $Q_{m+1} = (Q_{m-1} \times Q_2)$. It contains four disjoint copies of Q_{m-1} as subgraphs: $(Q_{m-1})^{00}$, $(Q_{m-1})^{01}$, $(Q_{m-1})^{10}$, $(Q_{m-1})^{11}$. We know $\gamma(Q_{m-1}) = \gamma_i(Q_{m-1}) = \gamma_p(Q_{m-1})$. Let D_1, D_2 be any two disjoint perfect dominating sets of $Q_{m-1,}$ which exist since by the standing hypothesis $m = 2^k$. Let $(D_1)^{00}$, $(D_2)^{11}$ be the corresponding dominating sets of $(Q_{m-1})^{00}$, $(Q_{m-1})^{11}$ respectively. Clearly, distance between $(Q_{m-1})^{00}$, $(Q_{m-1})^{11}$ is two, and distance between $(D_1)^{00}$, $(D_2)^{11}$ is three. Therefore, $(D_1)^{00} \cup (D_2)^{11}$ is an independent, perfect subset of $V(Q_{m+1})$. Similarly, $(D_1)^{01} \cup (D_2)^{10}$ is an independent perfect subset of $V(Q_{m+1})$. Note that neither of these is a dominating set since $m+1 = 2^k+1 \neq 2^q - 1$ for any q.

3.Domination of Q_n when $n = 2^k$

Theorem 3. 1 If $m = 2^k$, then $\gamma(Q_m) = 2^{m-k}$.

Proof: We have $Q_m = Q_{m-1} \times K_2$

Let Q_{m-1}^0, Q_{m-1}^1 be two copies of Q_{m-1}. Then $V(Q_m) = V(Q_{m-1}^0) \cup V(Q_{m-1}^1)$ and $E(Q_m) = E(Q_{m-1}^0) \cup E(Q_{m-1}^1) \cup \{u_i^0 u_i^1 : u_i \in V(Q_{m-1}) \text{ and } i = 1, 2, \ldots, t\}$ where $t = 2^{m-1}$

We know that $\gamma(Q_{m-1}) = \gamma_p(Q_{m-1}) = 2^{m-k-1}$.

Therefore, $\gamma(Q_m) \leq 2\gamma(Q_{m-1}) = 2^{m-k}$. ------- ------------------- (I)

Claim: $\gamma(Q_m) \geq 2^{m-k}$.

Let $D = S^0 \cup S^1$ be a minimum dominating set of Q_m where, $S^0 \subseteq V(Q_{m-1}^0)$ and $S^1 \subseteq V(Q_{m-1}^1)$.

Case 1: S^0 is a minimum dominating set of Q_{m-1}^0.

$|S^0| = 2^{m-k-1}$ and distance between any two vertices of S^0 is greater than or equal to 3. S^0 dominate all the vertices of Q_{m-1}^0 and 2^{m-k-1} vertices of Q_{m-1}^1. Hence S^0 dominates $2^{m-1} + 2^{m-k-1}$ vertices of Q_m.

Let $S^0 = \{v_i^0 / \ i = 1, 2, \ldots, 2^{m-k-1}, v_i \in V(Q_{m-1}) \text{ and } d(v_i, v_j) \geq 3\}$. Let $S = \{v_i^1 / \ i = 1,2,\ldots 2^{m-k-1}, v_i^0 \in S^0\} \subseteq V(Q_{m-1}^1)$. Then S^0 dominates Q_{m-1}^0 and S. $V(Q_{m-1}^1) - S$ contains $2^{m-1} - 2^{m-k-1}$ vertices each one is of degree m−2 in $V(Q_{m-1}^1) - S$. Hence minimum number of vertices needed to dominate these vertices are $(2^{m-1} - 2^{m-k-1})/(m-1) = (2^{m-k-1}(2^k-1))/(m-1)$

$= 2^{m-k-1}$, since $m = 2^k$.

Hence $|S^1| \geq 2^{m-k-1}$. So, $|D| = |S^0| + |S^1| \geq 2.\ 2^{m-k-1} = 2^{m-k}$

Hence $\gamma(Q_m) \geq 2^{m-k}$

Note: If S^1 is a minimum dominating set of $Q_{m-1}{}^1$ then we can prove $|S^0| \geq 2^{m-k-1}$.

Case 2(a): S^0, S^1 are not minimum dominating sets of $Q_{m-1}{}^0$, $Q_{m-1}{}^1$ respectively , but $|S^0| = 2^{m-k-1}$.

S^0 is not a minimum dominating set of $Q_{m-1}{}^0$ and hence it is not a dominating set of $Q_{m-1}{}^0$. That is S^0 is not dominating $Q_{m-1}{}^0$ and $|S^0| = 2^{m-k-1}$. This implies that there exist at least (m–2) elements of $Q_{m-1}{}^0$ which are not dominated by S^0.

To dominate these vertices in Q_m, (m–2) vertices must be included in S^1. These (m–2) vertices dominate at most (m–1)(m–2) other vertices in $Q_{m-1}{}^1$. Hence remaining vertices in $Q_{m-1}{}^1 = 2^{m-1} - (m-2) - (m-1)(m-2) = 2^{m-1} - (m-2)m$

To dominate these vertices at least $(2^{m-1} - (m(m-2)))/m$ vertices must be included in S^1.

Hence $|S^1| \geq (m-2) + (2^{m-1} - (m(m-2)))/m$

$= (2^{m-1})/m = 2^{m-k-1}$

Hence $|D| = |S^0| + |S^1| \geq 2.\ 2^{m-k-1} = 2^{m-k}$

$\gamma(Q_m) \geq 2^{m-k}$

<u>**Case 2(b):**</u> S^0, S^1 are not minimum dominating sets of $Q_{m-1}{}^0$, $Q_{m-1}{}^1$ respectively and either $|S^0| < 2^{m-k-1}$ or $|S^1| < 2^{m-k-1}$.

Let us assume that $|S^0| < 2^{m-k-1}$.

<u>**Subcase 1:**</u> $|S^0| = 2^{m-k-1} - 1$.

Since $\gamma(Q_{m-1}) = \gamma_p(Q_{m-1}) = 2^{m-k-1}$, there exist at least one $u^0 \in V(Q_{m-1}{}^0)$ such that elements of $N[u^0]$ is not dominated by any vertex of S^0 and $|N[u^0]| = m$.

So to dominate these vertices of $N[u^0]$ in Q_m, m vertices of $V(Q_{m-1}{}^1)$ must be included in D and hence in S^1.

These vertices are nothing but u^1 and neighbours of u^1 in $Q_{m-1}{}^1$.

These m elements dominate at most (m–1)(m–2) vertices of $Q_{m-1}{}^1$ other than elements of $N[u^1]$. (u^1dominate (m–1) elements which are elements of $N[u^1]$ only and an element $v \in N[u^1]$, $v \neq u^1$ dominate (m–2) elements which are not in $N[u^1]$).

So, remaining vertices of $Q_{m-1}{}^1$ is $2^{m-1} - m - (m-1)(m-2)$

$= 2^{m-1} - m - m^2 + 3m - 2$

$= 2^{m-1} - m^2 + 2m - 2.$

To dominate these vertices, at least $(2^{m-1} - m^2 + 2m - 2)/m$ vertices of $Q_{m-1}{}^1$ are needed.

Therefore, $|S^1| \geq m + (2^{m-1} - m^2 + 2m - 2)/m$

$= (2^{m-1})/m + 2 - 2/m$

$= (2^{m-1})/m + 1 + (1 - 2/m)$

$\geq 2^{m-k-1}+1$.

Hence, $|D| = |S^0| + |S^1| \geq (2^{m-k-1}-1) + (2^{m-k-1}+1)$

$= 2.\ 2^{m-k-1} = 2^{m-k}$

Therefore $\gamma(Q_m) \geq 2^{m-k}$.

Subcase 2: (general case) $|S^0| = 2^{m-k-1} - r,\ r > 0$

We know $\gamma(Q_{m-1}) = 2^{m-k-1}$ and any minimum dominating set of Q_{m-1} contain 2^{m-k-1} elements and is also perfect.

So, $|S^0| = 2^{m-k-1} - r$ implies that, there exists at least r elements $u_1^0, u_2^0, \ldots, u_r^0$ in $V(Q_{m-1}^0)$ with $d(u_i^0, u_j^0) \geq 3$, such that elements of $N[u_1^0] \cup N[u_2^0] \ldots \cup N[u_r^0]$ are not dominated by any element of S^0 in Q_{m-1}^0. So to dominate these vertices in Q_m, rm vertices of $N[u_1^0] \cup N[u_2^0] \ldots \cup N[u_r^0]$ must be included in D and hence in S^1.

These rm elements dominate at most r(m−1)(m−2) vertices of Q_{m-1}^1 other than the elements of $N[u_1^1] \cup N[u_2^1] \ldots \cup N[u_r^1]$.

So remaining vertices in Q_{m-1}^1 are $2^{m-1} - rm - r(m-1)(m-2)$

$= 2^{m-1} - rm^2 + 3rm - rm - 2r = 2^{m-1} - rm^2 + 2rm - 2r$.

To dominate these vertices at least $(2^{m-1} - rm^2 + 2rm - 2r)/m$ vertices are needed.

Hence $|S^1| \geq rm + (2^{m-1})/m - rm + 2r(m-1)/m$

$= 2^{m-k-1} + r(2-2/m) = 2^{m-k-1} + r(1+1-2/m) \geq 2^{m-k-1} + r$

Hence in this case also, $|D| = |S^0| + |S^1| \geq (2^{m-k-1} - r) + (2^{m-k-1} + r) = 2.\ 2^{m-k-1} = 2^{m-k}$

Therefore, $\gamma(Q_m) \geq 2^{m-k}$.

Hence in all cases $\gamma(Q_m) \geq 2^{m-k}$. -------------------------- (II)

From (I) and (II), $\gamma(Q_m) = 2^{m-k}$.

Note: Let D_1, D_2 be two disjoint perfect dominating sets of Q_{n-1}. Let D_1^0, D_2^1 be the corresponding dominating sets of $(Q_{n-1})^0$, $(Q_{n-1})^1$. Then $D_1^0 \cup D_2^1$ is an independent minimum dominating set for Q_n with cardinality 2^{n-k}.

Theorem 3.2 $\gamma_t(Q_m) = 2^{m-k} = \gamma_p(Q_m) = \gamma_i(Q_m)$, where $m = 2^k$.

Proof: $Q_m = Q_{m-1} \times K_2$. We can view Q_m as $V(Q_m) = V((Q_{m-1})^0) \cup V((Q_{m-1})^1)$.

$E(Q_m) = E((Q_{m-1})^0) \cup E((Q_{m-1})^1) \cup \{ u_i^0 u_i^1 : u_i^0 \in V((Q_{m-1})^0), u_i^1 \in V((Q_{m-1})^1)\}$, i =1, 2, …, m−1. Let D be a γ-dominating set of Q_{m-1}. Let D^0, D^1 be the corresponding dominating sets of Q_{m-1}^0, Q_{m-1}^1 respectively. $D^0 \cup D^1$ is a γ-dominating set of Q_m and is total. Therefore, $\gamma_t(Q_m) = 2 \times 2^{m-k-1} = 2^{m-k}$. We know that, $V(Q_{m-1})$ can be partitioned into 2^k sets each of which is the dominating set of Q_{m-1} containing 2^{m-k-1} elements and each of those dominating set is perfect in Q_{m-1}. Let $(D_1)^0$ and $(D_2)^0$ be two disjoint dominating sets in the domatic partition of $(Q_{m-1})^0$. Let $(D_2)^1 = \{u_i^1 : u_i^0 \in (D_2)^0\}$ then $(D_2)^1$ is a dominating set of $(Q_{m-1})^1$ and $(D_1)^0 \cup (D_2)^1$ is a minimum dominating set of Q_m and is independent. Therefore, $\gamma_i(Q_m) = 2 \times 2^{m-k-1} = 2^{m-k}$. This proves the theorem.

Theorem 3.3 If $n = 2^k+s$ with $0 < s < 2^k-1$, then $2^{n-k-1} < \gamma(Q_n) \leq 2^{n-k}$.

Proof: $n = 2^k+s$, $0 < s < 2^k-1$. We have $\gamma(Q_n) \leq 2\gamma(Q_{n-1})$ for all n. As $m = 2^k$, we have $\gamma(Q_{m+s}) \leq 2\gamma(Q_{m+s-1}) \leq 2^2\gamma(Q_{m+s-2}) \leq \ldots \ldots \leq 2^s\gamma(Q_m) = 2^s \times 2^{m-k} = 2^{m+s-k} = 2^{n-k}$. Also, $\gamma(Q_n) \geq 2^n/(n+1)$.

Hence, $\gamma(Q_{m+s}) \geq 2^{m+s}/(m+s+1) > 2^{m+s}/(2m) = 2^{m+s}/(2^{k+1}) = 2^{m+s-k-1} = 2^{n-k-1}$. Therefore, $2^{n-k-1} < \gamma(Q_n) \leq 2^{n-k}$. But $s > 0$ and $s < 2^k-1$. Hence, $2^{n-k-1} < \gamma(Q_n) \leq 2^{n-k}$.

4. Independent Domination of Q_n

Theorem 4.1 $\gamma_i(Q_n) \leq 2^{n-k}$ for all n such that $2^k-1 \leq n \leq 2^{k+1}-2$.

Proof: When $n = 2^k-1 = m-1$, we know $\gamma(Q_{m-1}) = \gamma_i(Q_{m-1}) = \gamma_p(Q_{m-1}) = 2^{m-k-1} = 2^{n-k}$ and $V(Q_{m-1})$ can be partitioned into $m = 2^k$ dominating sets of cardinality 2^{m-k-1}. Therefore, $V(Q_{m-1}) = D_1 \cup D_2 \cup \ldots \cup D_m$, where each D_i is a perfect dominating set for Q_{m-1} and $D_i \cap D_j = \varnothing$ for $i \neq j$. Let $(D_1)^0$, $(D_2)^1$ be dominating sets of $Q_{m-1}{}^0$ and $Q_{m-1}{}^1$ respectively corresponding to the dominating sets D_1, D_2 of Q_{m-1}.

In general, if $n = 2^k+s$ for $0 \leq s \leq 2^k-2$, then Q_n has a vertex partition into 2^{s+1}copies of Q_{m-1} and the degree of each vertex in Q_n is $n = 2^k+s = m+s$. Therefore only s+1 vertex disjoint copies of Q_{m-1} are at distance one from a particular vertex x of Q_n. Hence we can take copies of an independent dominating set of Q_{m-1} in such a way that their union is an independent dominating set for Q_{m+s}. (since $s \leq 2^k-2 = m-2$ and there are 2^k independent mutually disjoint dominating sets for Q_{m-1}, this is always possible). Thus, $\gamma_i(Q_n) = \gamma_i(Q_{m+s}) \leq 2^{s+1}\gamma_i(Q_{m-1}) = 2^{s+1}2^{m-k-1} = 2^{n-k}$; that is, $\gamma_i(Q_n) \leq 2^{n-k}$ for all n such that $2^k-1 \leq n \leq 2^{k+1}-2$.

5.Connected and Total domination of Q_n

Theorem 5.1

If $2^k-1 \leq n \leq 2^{k+1}-2$, then $\gamma_c(Q_n) \leq 2^{n-k} + 2^{m-k} - 2$, where $m = 2^k$.

Proof:

Case 1: $n = 2^k-1 = m - 1$.

Let D be a γ-dominating set of Q_{m-1}. We know $\gamma = \gamma_i = \gamma_p = |D| = 2^{m-k-1}$. The distance between any two elements in D is at least 3 and if $u \in D$, then there exists $v \in D$ such that $d(u, v) = 3$ in Q_{m-1}. Let $D = \{v_1, v_2, v_3, \ldots, v_t\}$, where $t = 2^{m-k-1}$.

Let D be the graph with vertex set D and whose edge set is defined by $< x, y > \in E(D)$ if and only if $d(x, y) = 3$ in Q_{m-1}. We claim that D is connected. If not V(D) can be partitioned into non-empty subsets D_1 and D_2 such that there is no edge in D between D_1 and D_2. This means that for any $x \in D_1$ and any $y \in D_2$, $d(x, y) \geq 4$ in Q_{m-1}. Let $d(D_1, D_2) = d(u_0, v_0) = d_0$, $u_0 \in D_1$, $v_0 \in D_2$. Let $u_0, x_1, x_2, \ldots, x_{d0} = v_0$ be a u_0, v_0 path of length d_0. If $x_2 \in N(u)$ for some $u \in D_1$ then $u, x_2, x_3, \ldots, x_{d0} = v_0$ is a path from D_1 to D_2 of length $d_0 -1$, contradicting the minimality of d_0. If $x_2 \in N(v)$ for some $v \in D_2$, then u_0, x_1, x_2, v is a D_1, D_2 path of length $3 \leq d(D_1, D_2)$ in Q_{m-1}. Thus $x_2 \notin N(D_1) \cup N(D_2) = N(D)$, contradicting the fact that D is a dominating set . Thus D must be connected. Hence it has a spanning tree, with $|D| -1$ edges. Now each edge $< x, y > \in D$ corresponds to an x, y path of length 3 in Q_{m-1}. So by adjoining to D two vertices per edge of D we obtain a connected dominating set S, of size $|D| +2(|D| -1) = (2^{m-k-1} +2\times 2^{m-k-1})-2 = (3\times 2^{m-k-1})-2 = 3\times 2^{n-k}-2$. This proves case 1.

Case 2: <u>$n = 2^k$</u> = m.

Let D be a dominating set of Q_{m-1}. Let $D^0 = \{x^0 \in Q_m \mid x \in D\}$ and $D^1 = \{x^1 \in Q_m \mid x \in D\}$, and for each $x \in D$, x^0 and x^1 are adjacent in Q_m. We have seen in the proof of part (1) above that there is a connected dominating set S of $(Q_{m-1})^0$ which contains D^0 and whose size is $(3\times 2^{m-k-1})-2$. Since each vertex of D^1 is adjacent to vertex of $D^0 \subseteq S$, $S\cup D^1$ is connected. Since for i = 0, 1 D^i dominates $(Q_{m-1})^i$, $D^0 \cup D_1$dominates Qm. Hence so does $S\cup D^1$, which is thus a connected dominating set for Q_m. Its size is $|S| + |D^1| = (3\times 2^{m-k-1})-2 +2^{m-k-1} = 2^2\times 2^{m-k-1}-2 = 2^{m-k+1}-2 = 2^{n-k+1}-2$. Thus $\gamma_c(Q_n) \leq 2^{n-k+1}-2$.

Case 3: $n = 2^k +i = m+i = (m-1) + (i+1)$

Consider Q_n, where $n = 2^k+i = m+i = (m-1)+(i+1)$, where $1 < i \leq 2^k-2$. Q_n has 2^{i+1} vertex disjoint copies of Q_{m-1}. Name them $(Q_{m-1})^x$, for each i+1 dimensional Boolean vector x. Let D be a perfect dominating set of Q_{m-1}. Let D^x be the corresponding dominating sets of $(Q_{m-1})^x$. As in the proof of case 1, we can find a connected dominating set D of $(Q_{m-1})^0$ with cardinality $(3\times 2^{m-k-1} -2)$. Therefore, $D \cup_x D^x$ is a connected dominating set of Q_n, where x ranges over all i+1 dimensional Boolean vectors.

Hence, $\gamma_c(Q_n) \leq (3\times 2^{m-k-1})-2 + \Sigma 2^{m-k-1} = (3\times 2^{m-k-1})-2 + (2^{i+1}-1)\, 2^{m-k-1}$
$= (2^{i+1}+2)\, 2^{m-k-1} -2 = [2^{n-k}+2^{m-k}]-2.$

Thus the result is true for $n = 2^k+i$, where $1 < i \leq 2^k-2$.

This proves the theorem.

Theorem 5.2

(1) If $n = 2^k-1$, then $\gamma_t(Q_n) \leq 2^{n-k+1}$.

(2) If $2^k \leq n \leq 2^{k+1}-2$, then $\gamma_t(Q_n) \leq 2^{n-k}$.

Proof of (1): $n = 2^k-1$.

Let D be a γ-dominating set of Q_n. Then $|D| = 2^{n-k} = \gamma(Q_n)$. Let $D' = D + e_1$, where e_1 = 100…0. Then $D \cup D'$ is a total dominating set and so $\gamma_t(Q_n) \leq |D \cup D'| = 2 \times |D| = 2^{n-k+1}$.

Proof of (2): $2^k \leq n \leq 2^{k+1}-2$.

Let $n = 2^k+i = m+i = (m-1)+(i+1)$, where $1 \leq i \leq 2^k-2$. Q_n has 2^{i+1} vertex disjoint copies of Q_{m-1}. Each can be denoted by $(Q_{m-1})^x$, where x is any i+1 dimensional Boolean vector. Let D be a perfect dominating set of Q_{m-1} and let D^x be the corresponding dominating sets of $(Q_{m-1})^x$. Therefore, $\cup_x D^x$ is a total dominating set of Q_n. Hence, $\gamma_t(Q_n) \leq 2^{i+1} \times 2^{m-k-1} = 2^{n-k}$. This completes the proof of the theorem.

Conjecture: We have $Q_n = Q_{n-1} \times K_2$. Any dominating set D of $Q_{n-1}{}^0$ dominates $|D|$ elements in $Q_{n-1}{}^1$ and the remaining vertices form a subgraph of $Q_{n-1}{}^1$ whose highest degree is n−2. Using the fact that the size of a minimum dominating set D of a graph G is bounded above by $\lceil |G| / (\Delta+1) \rceil$, we conjecture that when $2^k+1 \leq n \leq 2^{k+1}-2$,

$\gamma(Q_n) = \gamma(Q_{n-1}) + \lceil (2^{n-1} - \gamma(Q_{n-1}))/ (n-1) \rceil$.

References:

[1] S.Arumugam and R.Kala, Domination parameters of Hypercubes, Journal of Indian Math. Soc.Vol 65,Nos.1–4(1998), 31–38.

[2] Buckley. F, Harary. F, Distance in graphs, Addison–Wesley, Publishing company (1990).

[3] Bhanumathi, M., (2004), A Study on some Structural properties of Graphs and some new Graph operations on Graphs, Thesis, Bharathidasan University, Tamil Nadu, India.

[4] R.W.Hamming, Error Detecting and Error Correcting codes, Bell Syst.Tech.J.,vol.26, No. 2, April 1950, 147-160.

[5] Teresa W.Haynes, Stephen T.Hedetniemi, Peter J.Slater, Fundamentals of Domination in graphs, Marcel Dekker,Inc.

[6] P.K.Jha, Hypercubes, median graphs and products of graphs, Some Algorithmic and Combinatorial results, Ph.D. Dissertation, Department of Computer science, Iowa State University,1990.

[7] Jaeun Lee, Independent Perfect domination sets in Cayley graphs, J. Graph Theory 37: 213-219, 2001.

[8] B.Zelinka, Domination numbers of cube graphs, Math Slovaca, 32(2), (1982), 117–1.