Complementary tree nil domination number of Cartesian Product of Graphs

S. Muthammai\(^1\) and *G. Ananthavalli\(^2\)

\(^1\) Alagappa Government Arts College, Karaikudi-630 003, India
\(^2\) Government Arts College for Women ( Autonomous ), Pudukkottai-622001, India

Email: muthammai.sivakami@gmail.com, *dv.ananthavalli@gmail.com

**Abstract:** A set \(D\) of a graph \(G = (V, E)\) is a dominating set, if every vertex in \(V(G) - D\) is adjacent to some vertex in \(D\). The domination number \(\gamma(G)\) of \(G\) is the minimum cardinality of a dominating set. A dominating set \(D\) is called a complementary tree nil dominating set, if the induced subgraph \(< V(G) - D >\) is a tree and also the set \(V(G) - D\) is not a dominating set. The minimum cardinality of a complementary tree nil dominating set is called the complementary tree nil domination number of \(G\) and is denoted by \(\gamma_{ctnd}(G)\). In this paper, complementary tree domination numbers of Cartesian product of some standard graphs are found.

**Key words:** Domination number, Complementary tree nil domination number, Cartesian product.

1. Introduction

Graphs discussed in this paper are finite, undirected and simple connected graphs. For a graph \(G\), let \(V(G)\) and \(E(G)\) denote its vertex set and edge set respectively. A graph \(G\) with \(p\) vertices and \(q\) edges is denoted by \(G(p, q)\). The concept of domination in graphs was introduced by Ore\([5]\). A set \(D \subseteq V(G)\) is said to be a dominating set of \(G\), if every vertex in \(V(G) - D\) is adjacent to some vertex in \(D\). The cardinality of a minimum dominating set in \(G\) is called the domination number of \(G\) and is denoted by \(\gamma(G)\). Muthammai, Bhanumathi and Vidhya\([5]\) introduced the concept of complementary tree dominating set. A dominating set \(D \subseteq V(G)\) is said to be a complementary tree dominating set (ctd-set), if the induced subgraph \(< V(G) - D >\) is a tree. The minimum cardinality of a ctd-set is called the complementary tree domination number of \(G\) and is denoted by \(\gamma_{ctd}(G)\). Any undefined terms in this paper may be found in Harary\([2]\).

The cartesian product of two graphs \(G_1\) and \(G_2\) is the graph, denoted by \(G_1 \times G_2\) with \(V(G_1 \times G_2) = V(G_1) \times V(G_2)\) (where \(\times\) denotes the cartesian product of sets) and two vertices \(u = (u_1, u_2)\) and \(v = (v_1, v_2)\) in \(V(G_1 \times G_2)\) are adjacent in \(G_1 \times G_2\) whenever \([u_1 = v_1\) and \((u_2, v_2) \in E(G_2)]\) or \([u_2 = v_2\) and \((u_1, v_1) \in E(G_1)]\). The corona \(G_1 \bigcirc G_2\) of two graphs \(G_1\) and \(G_2\) are defined as the graph \(G\) obtained by taking one copy of \(G_1\) of order \(p_1\) and \(p_1\) copies of \(G_2\) and then joining the \(i^{th}\) vertex of \(G_1\) to every vertex in the \(i^{th}\) copy of \(G_2\). The Corona \(G_1 \bigcirc G_2\) has \(p_1(1 + p_2)\) vertices and \(q_1 + p_1q_2 + p_1p_2\) edges. The concept of complementary tree nil dominating set is introduced in \([4]\). A dominating set \(D \subseteq V(G)\) is said to be a
complementary tree nil dominating set (ctnd-set), if the induced subgraph <V(G) - D> is a tree and the set V(G) - D is not a dominating set. The minimum cardinality of a ctnd-set is called the complementary tree nil domination number of G and is denoted by γ_{ctnd}(G).

In this paper, we find an upper bound for complementary tree nil domination number of Cartesian product of P\textsubscript{m} \times P\textsubscript{n} and this number found for K\textsubscript{m} \times K\textsubscript{n}, K\textsubscript{m} \times P\textsubscript{n}, K\textsubscript{m} \times C\textsubscript{n} and C\textsubscript{m} \times P\textsubscript{n}.

2. Main Results

Theorem 2.1:

If G = K\textsubscript{m} \times K\textsubscript{n} (m,n \geq 3 and m \leq n), then γ_{ctnd}(G) = \begin{cases} (m(n - 2) + 3, & \text{if } m = n \\ (m(n - 2) + 2, & \text{if } m < n \end{cases}

Proof:

Let G = K\textsubscript{m} \times K\textsubscript{n}.

Let V(G) = \bigcup_{i=1}^{m} \{v_{i1}, v_{i2}, ..., v_{im}\} such that <\{v_{i1}, v_{i2}, ..., v_{im}\} \cong K\textsubscript{i}^1, i = 1, 2, ..., m and <\{v_{ij}, v_{ij+1}, ..., v_{mj}\} \cong K\textsubscript{j}^m, j = 1, 2, ..., n, where K\textsubscript{i} is the i\textsuperscript{th} copy of K\textsubscript{n} and K\textsubscript{j} is the j\textsuperscript{th} copy of K\textsubscript{m} in K\textsubscript{m} \times K\textsubscript{n}. |V(G)| = mn.

Case 1: m = n.

Let D' = \bigcup_{i=1}^{m-1} \{v_{ii}, v_{ii+1}\} \cup \{v_{mm}, v_{mn}\} and D = V(G) - D'. Then V(G) - D = D' and |D'| = 2(m - 2) + 1 = 2m - 3. The vertices v_{ii}, v_{ii+1} in V(G) - D are adjacent to v_{ii} in D, i = 2, 3, ..., m-1 and the vertex v_{mm} is adjacent to v_{mn} in D. Therefore D is a dominating set of G. Also <V(G) - D > \cong P_{2(m-2) + 1} = P_{2m-3}. Therefore D is a ctnd-set of G and since N(v_{ii}) \subseteq D, D is a ctnd-set of G. Therefore γ_{ctnd}(G) \leq |D'| = mn - (2m - 3) = m(n - 2) + 3.

It is to be noted that, any tree in G is a path and \delta(G) = m. Let D' be a γ_{ctnd}-set of G. Then there exists a vertex u \in D' such that N(u) \subseteq D'. The longest path that can be obtained from the subgraph of G induced by the vertices of V(G) - N(u) is P_{2m-3}. Therefore <V(G) - D'> \cong P_{2m-3}.

Therefore D' contains atleast mn - (2m - 3) = m(n - 2) + 3 vertices. Therefore γ_{ctnd}(G) = |D'| \geq m(n - 2) + 3.

Hence γ_{ctnd}(G) = m(n - 2) + 3.

Case 2: m < n.

Let D' = \bigcup_{i=2}^{m} \{v_{ii}, v_{ii+1}\} and D = V(G) - D'. Then V(G) - D = D' and |D'| = 2(m - 1). The vertices v_{ii}, v_{ii+1} (i = 2, 3, ..., m) are adjacent to v_{ii}, (i = 2, 3, ..., m) in D. Therefore D is a dominating set of G. Also <V(G) - D > \cong P_{2(m-2)} = P_{2m-2}. Therefore D is a ctnd-set of G and since N(v_{ii}) \subseteq D, D is a ctnd-set of G.

Therefore γ_{ctnd}(G) \leq |V(G)| - |D'| = mn - (2m - 2) = m(n - 2) + 2.
As in case 1, any tree in $G$ is a path and $\delta(G) = m$. Let $D'$ be a $\gamma_{ctnd}$-set of $G$. Then there exists a vertex $u \in D'$ such that $N(u) \subseteq D'$. The longest path that can be obtained from the subgraph of $G$ induced by the vertices of $V(G) - N(u)$ is $P_{2m - 2}$.

Therefore, $\langle V(G) - D' \rangle \cong P_{2m - 2}$, and $D'$ contains at least $mn - (2m - 2) = m(n - 2) + 2$ vertices. Therefore, $\gamma_{ctnd}(G) = |D'| \geq m(n - 2) + 2$.

Therefore, $\gamma_{ctnd}(G) = m(n - 2) + 2$.

Hence, $\gamma_{ctnd}(G) = \begin{cases} m(n - 2) + 3, & \text{if } m = n \\ m(n - 2) + 2, & \text{if } m < n \end{cases}$

**Example 2.1:**

For the graph $G$ given in Figure 1.a and Figure 1.b, the set of vertices within the is a minimum $ctnd$-set of $K_m \times K_n$, and $\gamma_{ctnd}(K_4 \times K_4) = 11$ and $\gamma_{ctnd}(K_4 \times K_5) = 14$.

![Figure 1.a](image1.png) ![Figure 1.b](image2.png)

**Theorem 2.2:**

If $G \cong K_m \times P_n$ ($4 \leq m \leq n$), then $\gamma_{ctnd}(G) = n(m - 2) + 2$.

**Proof:**

Let $G \cong K_m \times P_n$.

Let $V(G) = \bigcup_{i=1}^{m} \{v_{1i}, v_{2i}, ..., v_{ni}\}$ such that $\langle \{v_{1i}, v_{2i}, ..., v_{ni}\} \rangle \cong K_n$, $i = 1, 2, ..., m$ and $\langle \{v_{1j}, v_{2j}, ..., v_{mj}\} \rangle \cong P_n$, $j = 1, 2, ..., n$, where $K_n^i$ is the $i$-th copy of $K_n$ and $P_n^j$ is the $j$-th copy of $P_n$ in $K_m \times P_n$.

Let $D' = \begin{cases} \bigcup_{i=2}^{n} \{v_{2i}\} \cup \bigcup_{i=1}^{\frac{n}{2}} \{v_{3,2i}, v_{1,2i+1}\}, & \text{if } n \text{ is odd} \\ \bigcup_{i=2}^{n} \{v_{2i}\} \cup \bigcup_{i=1}^{\frac{n}{2}} \{v_{1,2i-1}, v_{3,2i}\}, & \text{if } n \text{ is even} \end{cases}$.

Then $|D'| = 2(n - 1)$. If $D = V(G) - D'$, then $D$ is a dominating set of $G$ and $N(v_{1i}) \subseteq D$. Also, $\langle V(G) - D \rangle = \langle D' \rangle \cong P_n \times K_i$. Therefore, $D$ is a $ctnd$-set of $G$.

$\gamma_{ctnd}(G) \leq |D| = mn - 2(n - 1) = mn - 2n + 2 = n(m - 2) + 2$.

Therefore, $\gamma_{ctnd}(G) \leq n(m - 2) + 2$.

Let $D'$ be a $\gamma_{ctnd}$-set of $G$. Since $D'$ is a $ctd$-set of $G$, $D'$ contains at least $(m - 2)$ vertices in each of $(n - 1)K_m$'s and since $V(G) - D'$ is not a dominating set, $D'$ contains...
all the vertices of the remaining $K_m$. Hence $D'$ contains at least $(m-2)(n-1)+m = mn-m-2n+2 + m = n(m-2) + 2$ vertices. Therefore $\gamma_{ctd}(G) = |D'| \geq n(m-2) + 3$.

Hence $\gamma_{ctd}(K_m \times P_n) = n(m-2) + 2$.

Example 2.2:

For the graph $G$ given in Figure 2, the set of vertices within the $\emptyset$ is a minimum $ctd$-set of $K_m \times K_n$ and $\gamma_{ctd}(K_4 \times K_9) = 20$.

Remark 2.1:

In view of Theorem 2.2, $\gamma_{ctd}(K_m \times C_n) = n(m-2) + 3$.

Theorem 2.3:

If $G \cong P_m \times P_n$ ($m, n \geq 2$), then $\gamma_{ctd}(G) \leq \gamma_{ctd}(G) + 2$.

Proof:

Let $G \cong P_m \times P_n$. Then $\delta(G) = 2$.

Let $D$ be a $\gamma_{ctd}$-set of $G$. Let $u \in D$ be a vertex of minimum degree in $G$ and $\deg(u) = \delta(G)$. Then $D' = D \cup N(u)$ is a $ctd$-set of $G$, since $N(u) \subseteq D'$. Therefore $\gamma_{ctd}(G) \leq |D'| = |D| + |N(u)| = \gamma_{ctd}(G) + \delta(G) = \gamma_{ctd}(G) + 2$.

Hence $\gamma_{ctd}(G) \leq \gamma_{ctd}(G) + 2$.

Equality holds, if $G \cong P_2 \times P_n$, $n \geq 3$.

Theorem 2.4:

If $G \cong C_3 \times P_n$, then $\gamma_{ctd}(G) = n + 2, n \geq 3$.

Proof:

Let $G \cong C_3 \times P_n$.

Let $V(G) = \bigcup_{i=1}^{n} \{v_{1i}, v_{2i}, v_{3i}\}$ such that $\langle \{v_{1i}, v_{2i}, \ldots, v_{1n}\} \rangle \cong P_i^n, i = 1, 2, 3$ and $\langle \{v_{1j}, v_{2j}, v_{3j}\} \rangle \cong C_3^n$, $j = 1, 2, \ldots, n$, where $P_i^n$ is the $i$th copy of $P_n$ and $C_3^n$ is the $j$th copy of $C_3$ in $C_3 \times P_n$. 
Let $D = \begin{cases} \{v_{11}, v_{21}\} \cup \bigcup_{i=1}^{n} \{v_{2i+1}, v_{3,2i-1}\}, & \text{if } n \text{ is even} \\ \{v_{11}, v_{21}, v_{31}\} \cup \bigcup_{i=1}^{n-1} \{v_{2i+1}, v_{3,2i+1}\}, & \text{if } n \text{ is odd.} \end{cases}$

Then $D$ is a dominating set of $G$ and $N(v_{11}) \subseteq D$. Also $\langle V(G) - D \rangle \cong P_n \circ K_1$.

Therefore $D$ is a ctnd-set of $G$.

$\gamma_{ctnd}(G) \leq |D| = \begin{cases} 2 \left( \frac{n}{2} \right) + 2 = n + 2, & \text{if } n \text{ is even} \\ 2 \left( \frac{n-1}{2} \right) + 3 = n + 2, & \text{if } n \text{ is odd.} \end{cases}$

Let $D'$ be a $\gamma_{ctnd}$-set of $G$. Then $D'$ contains at least one vertex from each cycle. Since $C_1 \times P_n$ contains $n$ copies of $C_1$, $D'$ contains at least $n$ vertices. Also, since $V(G) - D'$ is not a dominating set, the remaining vertices of first cycle $C_3$ in $C_3 \times P_n$ must be included in $D'$.

Therefore $D'$ contains at least $n + 2$ vertices and $\gamma_{ctnd}(G) = |D'| \geq n + 2$.

Hence $\gamma_{ctnd}(C_3 \times P_n) = n + 2, n \geq 3$.

**Theorem 2.5:**

If $G \cong C_4 \times P_n$, then $\gamma_{ctnd}(G) = \lfloor \frac{3n+4}{2} \rfloor, n \geq 2$.

**Proof:**

Let $G \cong C_4 \times P_n$ and $V(G) = \bigcup_{i=1}^{n} \{v_{1i}, v_{2i}, v_{3i}, v_{4i}\}$ such that $\langle v_{1i}, v_{2i}, \ldots, v_{ni} \rangle \cong P_i, i = 1, 2, 3, 4$ and $\langle v_{1j}, v_{2j}, v_{3j}, v_{4j} \rangle \cong C_4, j = 1, 2, \ldots, n$, where $P_i$ is the $i$th copy of $P_n$ and $C_4$ is the $j$th copy of $C_4$ in $C_4 \times P_n$ and $|V(G)| = 4n$.

**Case 1:** $n$ is even.

Let $D' = \{v_{31}, v_{3n}\} \cup \bigcup_{i=1}^{n-2} \{v_{1,2i+1}, v_{4,2i+1}, v_{3,2i+1}\} \cup \bigcup_{i=1}^{n-1} \{v_{2i+1}\}$ and $D = V(G) - D'$. Then $|D'| = 2 + 3 \left( \frac{n-2}{2} \right) + n - 1 = \frac{5n-4}{2}$. Then $D$ is a dominating set of $G$ and $N(v_{11}) \subseteq D$. Also $\langle V(G) - D \rangle = \langle D' \rangle$ is a tree obtained from a path $P_{n-1} = \langle v_{2j}, j = 1, 2, \ldots, n \rangle, (n \geq 2)$ by attaching $P_3$ at each of the vertices $v_{22}, v_{23}, v_{25}, \ldots, v_{2, n-1}$ and attaching a pendant edge at each of the vertices $v_{24}, v_{26}, \ldots, v_{2,n}$. Therefore $D$ is a ctnd-set of $G$.

$\gamma_{ctnd}(G) \leq |D| = |V(G) - D'| = 4n - \left( \frac{5n-4}{2} \right) = \frac{3n+4}{2}$.

Hence $\gamma_{ctnd}(G) \leq \frac{3n+4}{2}$.

Let $D'$ be a $\gamma_{ctnd}$-set of $G$. Since $\langle V(G) - D' \rangle$ is not a dominating set, $D'$ contains a vertex $u$ such that $N(u) \subseteq D$. $u$ is taken to be a vertex of minimum degree $\delta(G) = 3$ in $G$. The blocks $A$, $B$, $C$ are constructed as given below.
G is obtained by concatenating the blocks A, \( B^{\frac{n-2}{2}} \) and C. That is, \( G \cong A B^{\frac{n-2}{2}} C \). The vertices with the symbol \( \Theta \) in each of the blocks represent the vertices that are to be included in \( D' \).

Therefore \( D' \) contains 3 vertices from block A and at least 3 vertices from each block \( B^{\frac{n-2}{2}} \) and 2 vertices from block C. Therefore \( \gamma_{\text{ctnd}}(G) = |D'| \geq 3 + 3 \left( \frac{n-2}{2} \right) + 2 = \frac{3n+4}{2} \).

and hence \( \gamma_{\text{ctnd}}(G) = \frac{3n+4}{2} \).

Case 2: \( n \) is odd.

Let \( D' = \{v_{31}\} \cup \left[ \bigcup_{i=1}^{\frac{n-2}{2}} \{v_{1,2i+1}, v_{4,2i+1}, v_{3,2i+1}\} \right] \cup \left[ \bigcup_{i=2}^{n} \{v_{2i}\} \right] \).

Then \( |D'| = 1 + 3 \left( \frac{n-1}{2} \right) + n - 1 = \frac{5n-3}{2} \) and \( D = V(G) - D' \). Then \( D \) is a dominating set of \( G \) and \( N(v_{i}) \subseteq D \). Also \( <V(G) - D> = <D'> \) is a tree obtained from a path \( P_{n-1} = \left\{ v_{2,1}, i = 2, 3, ..., n \right\} \), \( (n \geq 2) \) by attaching \( P_{3} \) at each of the vertices \( v_{22}, v_{23}, v_{25}, ..., v_{2n} \) and attaching a pendant edge at each of the vertices \( v_{24}, v_{26}, ..., v_{2, n-1} \). Therefore \( D \) is a ctn-set of \( G \).

\[ \gamma_{\text{ctnd}}(G) \leq |D| = |V(G) - D'| = 4n - \left( \frac{5n-3}{2} \right) = \frac{3n+3}{2} \]  

Hence \( \gamma_{\text{ctnd}}(G) \leq \frac{3n+3}{2} = \left[ \frac{3n+4}{2} \right] \).

Let \( D' \) be a \( \gamma_{\text{ctnd}} \)-set of \( G \). Since \( <V(G) - D'> \) is not a dominating set, \( D' \) contains a vertex \( u \) such that \( N(u) \subseteq D \). \( u \) is taken to be a vertex of minimum degree \( \delta(G) = 3 \) in \( G \). The blocks A, B are constructed as in case 1.
Complementary tree nil domination number of Cartesian Product of Graphs

G is obtained by concatenating the blocks A and $B^{\frac{n-1}{2}}$ as in case 1. That is, $G \cong A^{\frac{n-1}{2}}B^{\frac{n-1}{2}}$. The vertices with the symbol $\bullet$ in each of the blocks represent the vertices that are to be included in $D'$. Therefore $D'$ contains 3 vertices from block A and at least 3 vertices from each block B of $B^{\frac{n-1}{2}}$.

Therefore $|D'| \geq 3 + 3 \left(\frac{n-1}{2}\right) = \frac{3n+3}{2} = \left\lceil\frac{3n+4}{2}\right\rceil$.

Hence $\gamma_{ctnd}(G) = \left\lceil\frac{3n+4}{2}\right\rceil$, $n \geq 2$.

**Theorem 2.6:**

If $G \cong C_5 \times P_n$, then $\gamma_{ctnd}(G) = 2n + 1$, $n \geq 3$.

**Proof:**

Let $G \cong C_5 \times P_n$ and $V(G) = \bigcup_{i=1}^{n} \{v_{1i}, v_{2i}, v_{3i}, v_{4i}, v_{5i}\}$ such that $\langle v_{1i}, v_{12}, \ldots, v_{1n} \rangle \cong P_n^i$, $i = 1, 2, 3, 4, 5$ and $\langle v_{1i}, v_{2i}, v_{3i}, v_{4i}, v_{5i} \rangle \cong C_5^j$, $j = 1, 2, \ldots, n$, where $P_n^i$ is the $i$th copy of $P_n$ and $C_5^j$ is the $j$th copy of $C_5$ in $C_5 \times P_n$. $|V(G)| = 5n$.

**Case 1:** $n$ is odd

Let $D = \{ v_{21}, v_{12}, v_{22} \} \cup \bigcup_{i=1}^{n+1} \{v_{1,2i-1}, v_{5,2i-1}\} \cup \bigcup_{i=2}^{n-1} \{v_{3,2i}, v_{4,2i}\}$. Then $|D| = 3 + 2 \left(\frac{n+1}{2}\right) + 2 \left(\frac{n-3}{2}\right) = 2n + 1$.

Consider the blocks

![Diagram](image.png)

**Figure 4**
Then $G \cong AB^{n-3}C$. Let $D$ be the set of vertices with the symbol $\bullet$ in each of the blocks $A$, $B^{n-2}$ and $C$. $D$ contains 5 vertices from block $A$, and 4 vertices from each block $B$ of $B^{n-2}$ and 2 vertices from block $C$. Then $D$ is a dominating set of $G$ and the vertex $v_{11}$ is such that $N(v_{11}) \subseteq D$ and $<V(G) - D>$ $\cong T$, where $T$ is a tree constructed as below.

Let $H$ be the graph obtained by subdividing each of the pendant edges of $P_{n-2}$ exactly once and $T$ be the tree obtained from $H$ by attaching a pendant edge at one pendant vertex say $v$ of $P_{n-2}$ and then joining a vertex of degree 2 of $P_i$ by an edge to a pendant vertex at a distance 2 from $v$.

Therefore $D$ is a ctnd-set of $G$.

$\gamma_{ctnd}(G) \leq |D| = 2n + 1.$

Let $D'$ be a $\gamma_{ctnd}$ set of $G$. Since $\gamma(C_5) = 2$, $D'$ contains 2 vertices from each of $n$ cycles and $D'$ contains one more vertex from a cycle $C_5$ and hence $D'$ contains at least 2n+1 vertices. Therefore $\gamma_{ctnd}(G) = |D'| \geq 2n + 1$.

Hence $\gamma_{ctnd}(G) = 2n + 1, n \geq 2$.

Case 2: $n$ is even

Let $D = \{v_{11}, v_{12}, v_{21}, v_{3}, v_{51}\} \cup \bigcup_{i=2}^{n} \{v_{1, 2i-1}, v_{3, 2i-1}, v_{4, 2i-1}, v_{5, 2i-1}\}$. Then $|D| = 5 + 4{\left(\frac{n-2}{2}\right)} = 2n + 1$.

$G$ is obtained by concatenating the blocks $A$, $B^{n-2}$. That is $G \cong AB^{n-2}$. Let $D$ be the set of vertices with the symbol $\bullet$ in each of the blocks $A$ and $B^{n-2}$. $D$ contains 5 vertices from block $A$, and 4 vertices from each block $B$ of $B^{n-2}$. Then $D$ is a dominating set of $G$ and the vertex $v_{11}$ is such that $N(v_{11}) \subseteq D$ and $<V(G) - D>$ $\cong T$, where $T$ is a tree constructed as in case 1.

Therefore $D$ is a ctnd-set of $G$ and $\gamma_{ctnd}(G) \leq |D| = 2n + 1$.

Let $D'$ be a $\gamma_{ctnd}$ set of $G$. Since $\gamma(C_5) = 2$, $D'$ contains 2 vertices from each of $n$ cycles and since $V(G) - D$ is not a dominating set of $G$, $D'$ contains one more vertex from a cycle $C_5$ and hence $D'$ contains at least 2n+1 vertices. Therefore $\gamma_{ctnd}(G) = |D'| \geq 2n + 1$.

Hence $\gamma_{ctnd}(G) = 2n + 1, n \geq 2$.

Theorem 2.7:

If $G \cong C_5 \times P_2$, then $\gamma_{ctnd}(G) = 5$. 
Proof:

Let $G \cong C_5 \times P_2$ and $V(G) = \bigcup_{i=1}^{5} \{v_{1i}, v_{2i}, v_{3i}, v_{4i}, v_{5i}\}$ such that $<\{v_{1i}, v_{2i}\}> \cong \mathbb{P}_6$, $i = 1, 2, 3, 4, 5$ and $<\{v_{1j}, v_{2j}, v_{3j}, v_{4j}, v_{5j}\}> \cong \mathbb{C}_6$, $j = 1, 2$, where $\mathbb{P}_6$ is the $i$th copy of $P_6$ and $\mathbb{C}_6$ is the $j$th copy of $C_6$ in $C_5 \times P_2$.

Let $D = \{v_{11}, v_{21}, v_{31}, v_{41}, v_{12}\}$. Then $N(V_{11}) \subseteq D$ and $D$ is a dominating set of $G$. Also $V(G) - D = \{v_{31}, v_{22}, v_{33}, v_{44}, v_{55}\}$ and $<V(G) - D>$ is a graph obtained from $P_3$ by attaching 2 pendant edges at a pendant vertex of $P_3$. Therefore $D$ is a ctnd-set of $G$.

$$\gamma_{ctnd}(G) \leq |D| = 5.$$

Let $D'$ be a $\gamma_{ctnd}$-set of $G$. $D'$ contains 4 vertices from $\mathbb{C}_6$ and at least one vertex from $\mathbb{C}_5$.

Therefore $D'$ contains at least 5 vertices. $\gamma_{ctnd}(G) = |D'| \geq 5$.

Hence $\gamma_{ctnd}(G) = 5$.

Theorem 2.8:

If $G \cong C_6 \times P_n$, then $\gamma_{ctnd}(G) = \frac{5n+1}{2}$, $n \geq 3$.

Proof:

Let $G \cong C_6 \times P_n$ and $V(G) = \bigcup_{i=1}^{n} \{v_{1i}, v_{2i}, v_{3i}, v_{4i}, v_{5i}, v_{6i}\}$ such that $<\{v_{1i}, v_{2i}, \ldots, v_{6n}\}> \cong \mathbb{P}_6$, $i = 1, 2, 3, 4, 5, 6$ and $<\{v_{1j}, v_{2j}, v_{3j}, v_{4j}, v_{5j}, v_{6j}\}> \cong \mathbb{C}_6$, $j = 1, 2, \ldots, n$, where $\mathbb{P}_6$ is the $i$th copy of $P_n$ and $\mathbb{C}_6$ is the $j$th copy of $C_6$. $|V(G)| = 6n$.

Case 1: $n$ is odd.

Let $D' = \{v_{31}, v_{41}, v_{51}, v_{32}, v_{52}, v_{62}\} \cup \bigcup_{i=1}^{2} \{v_{1i+1,1}, v_{5i+1,1}, v_{6i+1,1}\}\bigcup \bigcup_{i=1,2} v_{2i}.$

Then $|D'| = 5 + 3 \left(\frac{n-1}{2}\right) + n - 1 + 2 \left(\frac{n-3}{2}\right) = \frac{7n-1}{2}$ and $D = V(G) - D'$. Then $D$ is a dominating set of $G$ and $N(v_{1i}) \subseteq D$. Also $<V(G) - D > = <D'>$ is a tree obtained from a path $P_{n-1} = \langle v_{2i}, i = 2, 3, \ldots, n \rangle > (n \geq 2)$ by attaching $P_3$ at each of the vertices $v_{23}, v_{25}, v_{27}, \ldots, v_{2n}$ and attaching $P_j$ at each of the vertices $v_{24}, v_{26}, \ldots, v_{2,n-1}$. Therefore $D$ is a ctnd-set of $G$.

$$\gamma_{ctnd}(G) \leq |D| = |V(G) - D'| = 6n - \left(\frac{7n-1}{2}\right) = \frac{5n+1}{2}.$$

Hence $\gamma_{ctnd}(G) \leq \frac{5n+1}{2}$.

Let $D'$ be a $\gamma_{ctnd}$-set of $G$. Since $<V(G) - D'>$ is not a dominating set. Therefore $D'$ contains a vertex of $u$ such that $N(u) \subseteq D$. $u$ is taken to be a vertex of minimum degree $\delta(G) = 3$ in $G$. The blocks $A, B, C$ are constructed as given below.
G is obtained by concatenating the blocks A, B \( \frac{n-3}{2} \) and C. That is, G \( \cong \) A B \( \frac{n-3}{2} \) C. The vertices with the symbol \( \circ \) in each of the blocks represent the vertices that are to be included in \( D' \). Therefore \( D' \) contains 6 vertices from block A and at least 5 vertices from each block B of B \( \frac{n-3}{2} \) and 2 vertices from block C. Therefore \( \gamma_{ctnd}(G) = |D'| \geq 6 + \frac{5(n-3)}{2} + 2 = \frac{5n+1}{2} \) and hence \( \gamma_{ctnd}(G) = \frac{5n+1}{2} \).

**Case 2:** \( n \) is even.

Let \( D' = \{v_{31}, v_{41}, v_{51}, v_{32}, v_{62}\} \cup \bigcup_{i=1}^{\frac{n-3}{2}} \{v_{1,2i+1}, v_{5,2i+1}, v_{6,2i+1}\} \cup \bigcup_{i=2}^{n} \{v_{3,2i}, v_{4,2i}\} \). Then \( |D'| = 5 + 3\left(\frac{n-2}{2}\right) + n - 1 + 2\left(\frac{n-2}{2}\right) = \frac{7n-2}{2} \) and D = \( V(G) \) - \( D' \). Then D is a dominating set of G and \( N(v_{11}) \subseteq D \). Also \( <V(G) > - D > = <D'> \) is a tree obtained from a path \( P_{n-1} = <\{v_{2,i} | i = 2, 3, ..., n\}> \) for \( n \geq 2 \) by attaching \( P_{1} \) at each of the vertices \( v_{23}, v_{25}, v_{27}, ..., v_{2,n-1} \) and attaching \( P_{3} \) at each of the vertices \( v_{24}, v_{26}, ..., v_{2,n} \). Therefore D is a ctd-set of G.

\[ \gamma_{ctnd}(G) \leq |D| = |V(G) - D'| = 6n - \left(\frac{7n-2}{2}\right) = \frac{5n+2}{2}. \]

Hence \( \gamma_{ctnd}(G) \leq \frac{5n+2}{2} \).

Let \( D' \) be a \( \gamma_{ctnd} \)-set of G. Since \( <V(G) - D' > \) is not a dominating set, \( D' \) contains a vertex of u such that \( N(u) \nsubseteq D \). u is taken to be a vertex of minimum degree \( \delta(G) = 3 \) in G. The blocks A, B are constructed as in case 1.
G is obtained by concatenating the blocks A and $B_{\frac{n-2}{2}}$. That is, $G \cong AB_{\frac{n-2}{2}}$. The vertices with the symbol $\bigotimes$ in each of the blocks represent the vertices that are to be included in $D'$.

Therefore $D'$ contains 6 vertices from block A and atleast 5 vertices from each block B of $B_{\frac{n-2}{2}}$. Therefore $\gamma_{ctnd}(G) = |D'| \geq 6 + 5\left(\frac{n-2}{2}\right) = \frac{5n+2}{2}$ and hence $\gamma_{ctnd}(G) = \frac{5n+2}{2} \geq 5$.

Hence $\gamma_{ctnd}(C_6 \times P_n) = \frac{5n+2}{2}, n \geq 2$.

**Theorem 2.9:**

If $G \cong C_6 \times P_n$, then $\gamma_{ctnd}(G) = 5$.

**Proof:**

$G \cong C_6 \times P_n$ and $V(G) = \bigcup_{i=1}^{n} \left\{ v_{1i}, v_{2i}, v_{3i}, v_{4i}, v_{5i}, v_{6i} \right\}$ such that $\{v_{1i}, v_{2i}, ..., v_{6i}\} \cong P_6$, $i = 1, 2, 3, 4, 5, 6$ and $\{v_{1j}, v_{2j}, v_{3j}, v_{4j}, v_{5j}, v_{6j}\} \cong C_6$.

Let $D = \{v_{11}, v_{21}, v_{61}, v_{12}, v_{42}\}$. Then $N(v_{11}) \supseteq D$ and $D$ is a dominating set of $G$. Also $V(G) - D = \{v_{31}, v_{41}, v_{51}, v_{22}, v_{32}, v_{44}, v_{52}, v_{62}\}$ and $V(G) - D \supseteq P_7$.

Therefore $D$ is a ctn-ctn-set of $G$. $D'$ is a $\gamma_{ctnd}(G)$-set of $G$.

Let $D'$ be a $\gamma_{ctnd}(G)$-set of $G$. $D'$ contains 3 vertices from $C_6^0$ and atleast 2 vertices from $C_6^2$.

Therefore $D'$ contains atleast 5 vertices. Therefore $\gamma_{ctnd}(G) = |D'| \geq 5$.

Hence $\gamma_{ctnd}(G) = 5$.

**Remark 2.2:**

In view of Theorem 2.4, Theorem 2.5, Theorem 2.6, and Theorem 2.8,

1. $\gamma_{ctnd}(C_3 \times C_n) = n+3$, $n \geq 3$.
2. $\gamma_{ctnd}(C_4 \times C_n) = \left\lceil \frac{3n+6}{2} \right\rceil$, $n \geq 3$.
3. $\gamma_{ctnd}(C_5 \times C_n) = 2n+3$, $n \geq 3$.
4. $\gamma_{ctnd}(C_6 \times C_n) = 3n$, $n \geq 3$.

**Remark 2.3:**

1. If $G_1 \cong K_m$ and $G_2 \cong K_n$ then $\gamma_{ctnd}(G_1 + G_2) = m + n$.
2. If $G_1$ and $G_2$ are any two non-complete connected graphs of order m and n respectively, with minimum degree at least two, then $\gamma_{ctnd}(G_1 + G_2) \leq m + n - 1$.

Equality holds, if $G_1 \cong K_m - e, G_2 \cong K_n - e$. 

3. For any two connected graphs $G_1$ and $G_2$ of order $m$ and $n$ respectively,
\[ \gamma_{ctnd}(G_1 \Box G_2) \leq m + n - 1. \] Equality holds, if $G_1 \cong P_2$ and $G_2 \cong nK_1$.

4. For any two nontrivial connected graphs $G_1$ and $G_2$ with the of order $m$ and $n$ respectively,
\[ \gamma_{ctnd}(G_1 \Box G_2) \leq m + n - 2. \] Equality holds, if $G_1 \cong P_2$ and $G_2 \cong C_3$.

References:

Authors’ Profile:

S. Muthammai received the M.Sc. and M.Phil degree in Mathematics from Madurai Kamaraj University, Madurai in 1982 and 1983 respectively and received the Ph.D. degree in Mathematics from Bharathidasan University, Tiruchirappalli in 2006. From 16th September 1985 to 12th October 2016, she has been with the Government Arts College for Women (Autonomous), Pudukkottai, Tamilnadu and she is currently the Principal(Retd.) for Alagappa Government Arts College, Karaikudi, Tamilnadu. Her main area of research is domination in Graph Theory.

Ananthavalli G was born in Aranthangi, India, in 1976. She received the B.Sc. degree in Mathematics from Madurai Kamaraj University, Madurai, India, in 1996, the M.Sc. degree in Applied Mathematics from Bharathidasan University, Tiruchirappalli, India, in 2000, the M.Phil. degree in Mathematics from Madurai Kamaraj University, Madurai, India, in 2002, the B.Ed, degree from IGNOU, New Delhi, India, in 2007 and the M.Ed. degree from PRIST University, Thanjavur, India, in 2010. She was cleared SET in 2016. She has nearly 12 years of teaching experience in various schools and colleges. She is pursuing research in the department of Mathematics at Government Arts College for Women (Autonomous), Pudukkottai, India. Her main area of research is domination in Graph Theory.