On the Boolean Function Graph

\[ B( K_p, NINC, L(G)) \] of a Graph

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Abstract: For any graph \( G \), let \( V(G) \) and \( E(G) \) denote the vertex set and edge set of \( G \) respectively. The Boolean function graph \( B( K_p, NINC, L(G)) \) of \( G \) is a graph with vertex set \( V(G) \cup E(G) \) and two vertices in \( B( K_p, NINC, L(G)) \) are adjacent if and only if they correspond to two nonadjacent edges of \( G \) or to a vertex and an edge not incident to it in \( G \), where \( L(G) \) is the line graph of \( G \). For brevity, this graph is denoted by \( B_3(G) \). In this paper, structural properties of \( B_3(G) \) including traversability and eccentricity properties are studied. Also the graphs \( G \) for which \( B_3(G) \) contains \( C_n \) for \( n \geq 4 \) are obtained. Further, decomposition of \( B_3(G) \) for some known graphs are given.

Key Word: Boolean Function Graph.

1. Introduction

Graphs discussed in this paper are undirected and simple graphs. For a graph \( G \),
let \( V(G) \) and \( E(G) \) denote its vertex set and edge set respectively. Eccentricity of a vertex \( u \in V(G) \) is defined as \( e_G(u) = \max \{d_G(u, v) : v \in V(G)\} \), where \( d_G(u, v) \) is the distance between \( u \) and \( v \) in \( G \). We denote the eccentricity of vertex \( v \) in \( G \) as \( e(v) \) and the distance between two vertices \( u, v \) in \( G \) as \( d(u, v) \). The minimum and maximum eccentricities are the radius and diameter of \( G \), denoted \( r(G) \) and \( diam(G) \) respectively. When \( diam(G) = r(G) \), \( G \) is called a self-centered graph with radius \( r \), equivalently \( G \) is \( r \)-self-centered. A vertex \( u \) is said to be an eccentric point of \( v \) in a graph \( G \), if \( d(u, v) = e(v) \). In general, \( u \) is called an eccentric point, if it is an eccentric point of some vertex. We also denote the \( i^{th} \) neighborhood of \( v \) as \( N_i(v) = \{u \in V(G) : d_G(u, v) = i\} \) and denote the cardinality of the set \( H \) as \( |H| \). If \( |N_{i_e}(v)| \) is \( m \) for each point \( v \in V(G) \), then \( G \) is called an \( m \)-eccentric point graph. If \( m = 2 \), we call the graph \( G \) as bi-eccentric point graph. A connected graph \( G \) is said to be geodetic, if a unique shortest path joins any two of its vertices.

Whitney[17] introduced the concept of the line graph \( L(G) \) of a given graph \( G \) in 1932. The first characterization of line graphs is due to Krausz. The Middle graph \( M(G) \) of
a graph G was introduced by Hamada and Yoshimura[5]. Chikkodimath and Sampathkumar[3] also studied it independently and they called it, the semi-total graph T1(G) of a graph G. Characterizations were presented for middle graphs of any graph, trees and complete graphs in [1]. The concept of total graphs was introduced by Behzad[2] in 1966. Sastry and Raju[16] introduced the concept of quasi-total graphs and they solved the graph equations for line graphs, middle graphs, total graphs and quasi-total graphs. These graphs are very much useful in the construction of various related networks from the underlying graphs of networks. This motivates us to define and study other graph operations. Using L(G), G, incident and non-incident, complementary operations, complete and totally disconnected structures, one can get thirty-two graph operations. As already total graphs, semi-total edge graphs, semi-total vertex graphs and quasi-total graphs and their complements (8 graphs) are defined and studied, we have studied all other similar remaining graph operations.

The points and lines of a graph are called its elements. Two elements of a graph are neighbors, if they are either incident or adjacent. The **Total graph** T(G) of G has vertex set V(G)∪E(G) and vertices of T(G) are adjacent, whenever they are neighbors in G. The **Quasi-total graph** P(G) of G is a graph with vertex set as that of T(G) and two vertices are adjacent if and only if they correspond to two nonadjacent vertices of G or to two adjacent edges of G or to a vertex and an edge incident to it in G. The **Middle graph** M(G) of G is one whose vertex set is as that of T(G) and two vertices are adjacent in M(G), whenever either they are adjacent edges of G or one is a vertex of G and the other is an edge of G incident with it. Clearly, E(M(G)) = E(T(G))−E(G).

The **Boolean function graph** B( Kp, NINC, L(G)) G is a graph with vertex set V(G)∪E(G) and two vertices in B( Kp, NINC, L(G)) are adjacent if and only if they correspond to two nonadjacent edges of G or to a vertex and an edge not incident to it in G. For brevity, this graph is denoted by B3(G), where L(G) is the line graph of G. The vertices of G and L(G) in B3(G) are referred as point and line vertices respectively. In this paper, the properties of the Boolean function graph B3(G) are studied. The line vertex in B3(G) corresponding to an edge e in G is denoted by e'.

### 2. Properties

In this section, properties of B3(G) including traversability, eccentricity properties are studied. Also the graphs G for which B3(G) contains Cn, for n ≥ 4 are obtained. Further, decomposition of B3(G) for some known graphs are given.
Observation 2.1.
1. $L(G)$ is an induced subgraph of $B_3(G)$ and subgraph of $B_3(G)$ induced by point vertices is totally disconnected.
2. If $d_i = \deg_G(v_i)$, $v_i \in V(G)$, then the number of edges in $B_3(G)$ is
   
   \[
   \frac{(q/2)(2p + q - 3)}{2} - \frac{1}{2} \sum_{1 \leq i \leq p} d_i^2.
   \]
3. The degree of a point vertex $v$ in $B_3(G)$ is $q - \deg_G(v)$ and the degree of a line vertex $e'$
   in $B_3(G)$ is $\deg_{L(G)}(e') + p - 2 = p + q - \deg_{L(G)}(e') - 3$ and hence
   
   $\delta(B_3(G)) = q - \Delta(G)$.
4. $B_3(G)$ contains isolated vertices if and only if $G$ is one of the following graphs. $K_2$, $K_{1,n}$, $n \geq 2$, $nK_1$, $K_2 \cup nK_1$, $K_{1,m} \cup nK_1$, where $n \geq 1$ and $m \geq 2$.
5. $B_3(G)$ is totally disconnected if and only if $G \cong nK_1$ or $K_2$, $n \geq 1$.
6. $B_3(G)$ is disconnected if and only if $G \cong nK_1$, $K_2 \cup mK_1$, $C_3 \cup mK_1$ and $K_{1,n} \cup mK_1$, for $m \geq 0$ and $n \geq 2$.
7. Let $G$ be any graph having at least one edge and $G \neq K_n$, $n \geq 2$. Then $B_3(G)$ is
   biregular if and only if $G$ is regular.
8. Both $G$ and $B_3(G)$ are regular if and only if either $G$ is totally connected or $G$ is complete.

Proposition 2.2.
For any $(p, q)$ graph $G$, $B_3(G)$ contains cut-vertices if and only if there exists a
vertex in $G$ of degree $q - 1$.
Proof:
Assume there exists a vertex $v \in V(G)$ such that $\deg_G(v) = q - 1$. Therefore, there exists
exactly one edge $e$ in $G$, which is not incident with $v$ and hence $\deg_{B_3(G)}(v) = 1$. Also,

$B_3(G) - e$ disconnected with an isolated vertex $v$ in $B_3(G)$ and hence $e'$ is a cut-vertex in $B_3(G)$.

Conversely, assume $B_3(G)$ contains cut-vertices. Suppose there exists no vertex $v \in V(G)$ such that $\deg_G(v) = q - 1$. Therefore, all the point vertices of $B_3(G)$ have degree at
least two. That is, $\deg_{B_3(G)}(v) \geq 2$, for all $v \in V(B_3(G))$ and each line vertex in $B_3(G)$ is
adjacent to exactly $(p - 2)$ point vertices and hence no vertex of $B_3(G)$ is a cut-vertex, which is a contradiction. Hence, there exists at least one vertex $v \in V(G)$ such that $\deg_G(v) = q - 1$. 


Proposition 2.3.

For any graph $G$ with at least five vertices and $\beta_1(G) \geq 2$, $B_3(G)$ contains triangles if and only if $G$ contains $2K_2 \cup K_1$ (with $2K_2$ induced) as a subgraph, where $\beta_1(G)$ is the line independence number of $G$.

Proof:

Assume $B_3(G)$ contains triangles. Since, neither $G$ nor $\overline{G}$ is a subgraph of $B_3(G)$, at least two vertices of the triangle must be line vertices.

(i) If the two vertices of the triangle are line vertices and the third vertex is a point vertex, then $G$ contains $2K_2 \cup K_1$ (with $2K_2$ induced) as a subgraph.

(ii) If all the vertices of the triangle are line vertices, then $\beta_1(G) \geq 3$.

Converse follows from the construction of $B_3(G)$.

In the following, we find the girth of $B_3(G)$.

Proposition 2.4.

If $G$ is a graph with at least four vertices, then the girth of $B_3(G)$ is 3, 4 or 5.

Proof:

Case (i): $\beta_1(G) \geq 2$

If $G$ contains at least five vertices, then by Proposition 2.3, $B_3(G)$ contains triangles. Assume $G$ contains exactly four vertices. Then, $B_3(G)$ contains either no cycles or contains $C_5$ as an induced subgraph.

Case (ii): $\beta_1(G) = 1$

If $G$ contains $P_3 \cup 2K_1$ as a subgraph, then $B_3(G)$ contains either $C_4$ as an induced subgraph or contains no cycles.

Hence, girth of $B_3(G)$ is 3, 4 or 5.

Remark 2.5.

If $G$ contains two or three vertices, then $B_3(G)$ is cycle-free.

Proposition 2.6.

Let $G$ be a connected graph with at least four vertices other than a star. Then $B_3(G)$ is geodetic if and only if $G$ is a path on four vertices.

Proof:

Case (i): $G$ contains at least five vertices

Since $G$ is connected, $G$ contains $P_3 \cup K_2$ as a subgraph and $B_3(G)$ contains $C_4$ as an induced subgraph and hence, $B_3(G)$ is not geodetic.

Case (ii): $G$ contains four vertices and is not a path on four vertices.
If $G$ is not a path on four vertices, then $B_3(G)$ contains $C_6$ as an induced subgraph and hence $B_3(G)$ is not geodetic.

By Case (i) and Case (ii), we see that $G$ is a path on four vertices. Conversely, if $G$ is a path on four vertices, then $B_3(G)$ is geodetic.

**Proposition 2.7.**

For any graph $G$ with at least four vertices, $B_3(G)$ contains $K_{1,3}$ as an induced subgraph if and only if either $\beta_1(G) \geq 2$ or $G$ contains $C_3 \cup K_1$ or $K_{1,3} \cup K_1$ as a subgraph.

**Proof:**

Assume $B_3(G)$ contains $K_{1,3}$ as an induced subgraph. If $\beta_1(G) = 1$ and if $G$ is star on four vertices, then $B_3(G)$ is isomorphic to $C_7 \cup K_1$. Hence, either $\beta_1(G) \geq 2$ or $G$ contains $C_3 \cup K_1$ or $K_{1,3} \cup K_1$ as a subgraph.

Conversely, if either $\beta_1(G) \geq 2$ or $G$ contains $C_3 \cup K_1$ or $K_{1,3} \cup K_1$ as a subgraph, then $B_3(G)$ contains $K_{1,3}$ as an induced subgraph.

In the following, a necessary and sufficient condition for $B_3(G)$ to be Eulerian is given. For simplicity, the degree of a vertex $v$ in $B_3(G)$ is denoted by $d_3(v)$.

**Theorem 2.8.**

Let $G$ be any $(p, q)$ graph such that $B_3(G)$ is connected. Then $B_3(G)$ is Eulerian if and only if one of the following holds.

(i). $p$ is odd, $q$ is even and $G$ or each component of $G$ is Eulerian; and

(ii). $q$ is odd and each vertex in $G$ is of odd degree.

**Proof.**

Assume $B_3(G)$ is Eulerian. Therefore, each vertex in $B_3(G)$ is of even degree. If $v$ is a point vertex in $B_3(G)$, then $d_3(v) = q - \deg_G(v)$ is even and hence $q$ and $\deg_G(v)$ are of same parity. Similarly, if $e'$ is a line vertex in $B_3(G)$, then $d_3(e') = \deg_{\overline{L}(G)}(e') + p - 2 = p + q - \deg_{\overline{L}(G)}(e') - 3$ is even. Therefore, if $q$ is even, then $p$ must be odd, since $\deg_{\overline{L}(G)}(e')$ is even. Assume $q$ is odd. Since $q$ and $\deg_G(v)$ are of same parity, $\deg_G(v)$ is odd. Since the number of odd degree vertices in $G$ is even, $p$ is even. Thus, condition (i) or (ii) holds. Converse follows easily.

In the following, a necessary condition that $B_3(G)$ to be Hamiltonian is given.

**Theorem 2.9.**

Let $G$ be any $(p, q)$ with $p \geq 5$ and $\Delta(G) \leq q - 2$. If $\Delta(G) \leq (q-p)/2$, then $B_3(G)$ is Hamiltonian.
Proof.

Let \( v \in V(G) \) be such that \( \deg_G(v) = \Delta(G) \). Then \( v \in V(B_3(G)) \). If \( d_i(v) \) is the degree of \( v \) in \( B_i(G) \), then \( d_i(v) = q - \Delta(G) = \delta(B_i(G)) \). Since \( \Delta(G) \leq (q - p)/2 \), \( \delta(B_i(G)) \geq q - ((q - p)/2) \) and hence \( \delta(B_i(G)) \geq (p + q)/2 \). Thus, \( B_i(G) \) is Hamiltonian.

Example 2.10.

(i). \( B_3(K_5) \cong \) Petersen graph and hence not Hamiltonian.

(ii). \( B_3(C_n), B_3(C_{m+}), B_3(P_n), B_3(W_t) \) and \( B_3(K_t) \) are Hamiltonian graphs, where \( n \geq 4, m \geq 3 \) and \( t \geq 5 \), where \( C_n \) is a cycle on \( n \) vertices, \( C_{m+} \) is the graph obtained from the cycle \( C_m \) by attaching exactly one edge at each of its vertices, \( P_n \) is a path on \( n \) vertices, \( W_t \) is a wheel and \( K_t \) is a complete graph on \( t \) vertices.

Remark 2.11.

If \( G \) is any \((p, q)\) graph other than a star and \( q = p - 1 \), then \( B_3(G) \) contains a Hamiltonian path.

In the following, the eccentricity properties of \( B_3(G) \) for any graph \( G \) are discussed. Here, the graphs \( G \) for which \( B_3(G) \) are connected are considered. First, the graph \( G \) for which \( B_3(G) \) is self-centered with radius 2 is characterized. For simplicity, the distance between two vertices \( u, v \) in \( B_3(G) \) and the eccentricity of a vertex \( v \) in \( B_3(G) \) are denoted by \( d_3(u, v) \) and \( e_3(v) \) respectively. Since, there is no vertex of degree \( p + q - 1 \) in \( B_3(G) \), radius of \( B_3(G) \) is at least 2.

Theorem 2.12.

Let \( G \) be any graph with at least two edges. Then \( B_3(G) \) is self-centered with radius 2 if and only if for every pair of vertices \( u, v \) in \( G \) there exists at least one edge not incident with both \( u \) and \( v \).

Proof.

Assume for every pair of vertices \( u, v \) in \( G \) there exist at least one edge not incident with both \( u \) and \( v \).

(i). Let \( v_1 \) and \( v_2 \) be two point vertices in \( B_3(G) \). Then \( v_1, v_2 \in V(G) \). By the assumption, there exists an edge in \( G \) not incident with both \( v_1 \) and \( v_2 \). Then, \( d_3(v_1, v_2) = 2 \).

(ii). Let \( v \) and \( e' \) be a point and line vertices in \( B_3(G) \) respectively and \( e \) be the edge in \( G \) corresponding to \( e' \). If \( e \) is not incident with \( v \) in \( G \), then \( d_3(v, e') = 1 \). Let \( e \in E(G) \) be incident with \( v \) in \( G \). By the assumption, there exists at least one edge in \( G \) not adjacent to \( e \). Then \( d_3(v, e') = 2 \).
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(iii). Let $e'_1$ and $e'_2$ be two line vertices in $B_3(G)$ and $e_1, e_2$ be the corresponding edges in $G$. If $e_1$ and $e_2$ are nonadjacent edges in $G$, since $\overline{\text{L}(G)}$ is an induced subgraph of $B_3(G)$, $d_3(e'_1, e'_2) = 1$. Let $e_1$ and $e_2$ be adjacent edges in $G$. By the assumption, there exists at least one vertex $v$ in $G$ not incident with both $e_1$ and $e_2$ and hence $d_3(e'_1, e'_2) = 2$. Hence, it follows that all the vertices in $B_3(G)$ have eccentricity 2 and is self-centered with radius 2.

Conversely, assume there exists a pair of vertices $u, v$ in $G$ such that each edge in $G$ is incident with at least one of $u$ and $v$. Then $u, v \in V(B_3(G))$. If $(u, v) \notin E(G)$, then $d_3(u, v) = 3$. Let $(u, v) \in E(G)$ and $e = (u, v)$ and let $e'$ be the corresponding line vertex in $B_3(G)$. Then $v, u, e' \in V(B_3(G))$ and $d_3(v, e') = d_3(u, e') = 3$, which is a contradiction.

From the above facts, following are immediate consequences.

**Remark 2.13.**

Let $G$ be any graph such that $B_3(G)$ is connected. Then $B_3(G)$ is bi-eccentric with radius 2 if and only if there exists at least one pair of vertices $u, v$ in $G$ such that each edge in $G$ is incident with at least one of $u$ and $v$. Hence, by Theorem 6.1.4, it is clear that the diameter of $B_3(G)$ is at most 3.

**Remark 2.14.**

If $G \cong K_{1,n}$, $n \geq 2$, then $B_3(G) \cong C \cup K_1$, where the component $C$ is self-centered with radius 3.

In the following, a necessary and sufficient condition that $B_3(G)$ contains $C_n (n \geq 4)$, as an induced subgraph is obtained, where $G$ is any graph not totally disconnected.

**Proposition 2.15.**

$B_3(G)$ contains $C_4$ as an induced subgraph if and only if $G$ contains $P_3 \cup 2K_1$ as a subgraph.

**Proof:**

Assume $B_3(G)$ contains $C_4$ as an induced subgraph. Since, no two point vertices in $B_3(G)$ are adjacent, any $C_4$ in $B_3(G)$ can have at most two point vertices. If two vertices of $C_4$ in $B_3(G)$ are line vertices and the remaining two vertices are point vertices, then $G$ contains $P_3 \cup 2K_1$ as an induced subgraph. The other cases also give rise to $P_3 \cup 2K_1$ as a subgraph. Converse follows easily.
Proposition 2.16.

\( B_3(G) \) contains \( C_5 \) as an induced subgraph if and only if \( G \) contains either \( C_5 \) or \( P_4 \) as a subgraph.

Proof:

Assume \( B_3(G) \) contains \( C_5 \) as an induced subgraph. A cycle on five vertices in \( B_3(G) \) is possible, if the cycle contains either three line vertices and two point vertices or all line vertices. That means, \( G \) contains either \( P_4 \) or \( C_5 \) as a subgraph.

Similarly, the following propositions can be proved.

Proposition 2.17.

\( B_3(G) \) contains \( C_6 \) as an induced subgraph if and only if \( G \) contains \( G_1 \) as an induced subgraph or \( G \) contains \( C_4 \) or \( K_{1,3} \) as a subgraph, where \( G_1 \) is the graph obtained from \( K_4 - \epsilon \) by subdividing its diagonal edge exactly once.

Proposition 2.18.

\( B_3(G) \) contains no \( C_n \), \( (n \geq 7) \) as an induced subgraph.

Remark 2.19.

From the above propositions, it follows that \( B_3(G) \) contains cycles if and only if \( G \) contains one of the following graphs as a subgraph. \( 2K_2 \bigcup K_1 \) (with \( 2K_2 \) induced), \( P_3 \bigcup 2K_1 \), \( C_5 \), \( P_4 \), \( C_4 \), \( K_{1,3} \) and the graph \( G_1 \), where \( G_1 \) is the graph obtained from \( K_4 - \epsilon \) by subdividing its diagonal edge exactly once.

In the following, the edge partition of \( B_3(G) \) for some known graphs are given.

Theorem 2.20.

Let \( G \) be any connected \((p, q)\) graph such that \( G \neq K_{1,n} \) and \( K_3 \). Then the edges of \( B_3(G) \) can be partitioned into \( L(G) \) and \( qK_{1,p-2} \), where the center vertex of each \( K_{1,p-2} \) is a line vertex.

Proof:

Follows from the construction of \( B_3(G) \).

Theorem 2.21.

The edges of \( B_3(C_n) \), \((n \geq 4)\) can be partitioned into \((n-2)/2 \) \( C_{2n} \), \((n-4)/2\) \( C_n \) and \((n/2) \) \( K_{p,2} \), if \( n \) is even; and \((n-3)/2 \) \( C_{2n} \), \((n-3)/2\) \( C_n \) and \( nK_{2,2} \), if \( n \) is odd.

Proof:

Edges of \( B_3(C_n) \) can be partitioned into \( L(C_n) \) and \( nK_{1,n-2} \). But \( L(C_n) \) is a \((n-3)\)-regular graph on \( n \) vertices. This can be partitioned into \((n-4)/2\) \( C_n \) and \((n/2)K_{2,2} \), if \( n \)
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is even; and $((n-3)/2)C_{2n}$ if $n$ is odd. The edges of $nK_{1,n-2}$ can be partitioned into $((n-2)/2)C_{2n}$ if $n$ is even; and $((n-3)/2)C_{2n}$ and $nK_2$, if $n$ is odd. Thus, the theorem follows.

**Theorem 2.22.**

$B_3(K_{1,n})$, $(n \geq 2)$ is disconnected with exactly two components, one of the components being $K_1$. If the other component is $C$, then the edges of $C$ can be partitioned into $((n-2)/2)C_{2n}$ and $nK_2$, if $n$ is even; and $((n-1)/2)C_{2n}$, if $n$ is odd.

**Proof:**

$B_3(K_{1,n}) = K_1 \cup C$, where $E(C) = E((n-1)K_{1,n-1})$. Edges of $(n-1)K_{1,n-1}$ can be partitioned into $((n-2)/2)C_{2n}$ and $nK_2$, if $n$ is even; and $((n-1)/2)C_{2n}$, if $n$ is odd.

**Theorem 2.23.**

The edge set of $B_3(K_n)$, $(n \geq 4)$ can be partitioned into $(n-1)/2$ times $(n-2)$-regular graph on $2n$ vertices and $((n-2)(n-3)/2)$-regular graph on $(n(n-1))/2$ vertices, if $n$ is odd; and $(n-2)/2$ times $(n-2)$-regular graph on $2n$ vertices, $(n/2)K_{1,n-2}$ and $((n-2)(n-3)/2)$-regular graph on $(n(n-1))/2$ vertices, if $n$ is even.

**Proof:**

Edges of $B_3(K_n)$ can be partitioned into $L(K_n)$ and $((n(n-1)/2)K_{1,n-2}$. But $L(K_n)$ is a $(n-2)(n-3)/2)$-regular graph on $(n(n-1)/2$ vertices. The edges of $((n(n-1)/2)K_{1,n-2}$ can be partitioned into $(n-1)/2$ times $(n-2)$-regular graph on $2n$ vertices, if $n$ is odd; and $(n-2)/2$ times $(n-2)$-regular graph on $2n$ vertices and $(n/2)K_{1,n-2}$, if $n$ is even. Thus, the theorem follows.

**References**


